Designing a Dynamic Output Feedback Controller: An Example

Consider the following state-space model:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

(a) Is this system controllable?

Solution. By inspection, we see that the state-space model is in CCF, so the system is controllable. However, we can also directly compute the controllability matrix:

$$\begin{aligned} \mathcal{C}(A,B) &= \big[B \,|\, AB\big] \\ AB &= \begin{pmatrix} 0 & 1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \\ \therefore\ \mathcal{C}(A,B) &= \begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix} \\ \det \mathcal{C}(A,B) &= -1 \qquad\qquad \text{– the system is controllable} \end{aligned}$$

(b) Is this system observable?

Solution. Let's compute the observability matrix:

$$\begin{aligned} \mathcal{O}(A,C) &= \left[\frac{C}{CA}\right] \\ CA &= \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 6 \end{pmatrix} \\ \therefore\ \mathcal{O}(A,C) &= \begin{pmatrix} 2 & 1 \\ 2 & 6 \end{pmatrix} \\ \det \mathcal{O}(A,C) &= 10 \qquad\qquad \text{– the system is observable} \end{aligned}$$

(c) Design an observer to place observer poles at -5 and -5.

Solution. To do this, let us convert the state-space model into OCF. Since the system is observable, there exists an invertible coordinate transformation T that does this. The transformed system will be

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$$
$$y = \bar{C}\bar{x}$$

where $\bar{x} = Tx$, $\bar{A} = TAT^{-1}$, $\bar{B} = TB$, and $\bar{C} = CT^{-1}$. Since for observer design we only care about A and C, we do not need to determine the new system in its entirety, just the new system matrix $\bar{A} = TAT^{-1}$ and the new output matrix $\bar{C} = CT^{-1}$.

A quick way to find $\bar{A}$ and $\bar{C}$ is to use the fact that the original system is in CCF. To pass from CCF to OCF, we simply take $\bar{A} = A^T$ and $\bar{C} = B^T$, which gives

$$\bar{A} = \begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix}, \qquad \bar{C} = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

However, we can solve for $\bar{A}$ and $\bar{C}$ directly. Since the new system will be in OCF, we know that $\bar{A}$ and $\bar{C}$ will have the form

$$\bar{A} = \begin{pmatrix} 0 & -a_2 \\ 1 & -a_1 \end{pmatrix}, \qquad \bar{C} = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

To determine the entries a_1 and a_2, we use the fact that the characteristic polynomials of A and $\bar{A}$ are the same:

$$\det(Is - A) = \det(Is - \bar{A})$$
$$\det\begin{pmatrix} s & -1 \\ -2 & s-4 \end{pmatrix} = \det\begin{pmatrix} s & a_2 \\ -1 & s+a_1 \end{pmatrix}$$
$$s(s-4) - 2 = s(s+a_1) + a_2$$
$$s^2 - 4s - 2 = s^2 + a_1 s + a_2$$

Matching coefficients, we get $a_1 = -4, a_2 = -2$, so the new system will have

$$\bar{A} = \begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix}, \qquad \bar{C} = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

Either way, we can compute the new observability matrix:

$$\mathcal{O}(\bar{A}, \bar{C}) = \left[\begin{array}{c} \bar{C} \\ \hline \bar{C}\bar{A} \end{array}\right]$$
$$\bar{C}\bar{A} = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 4 \end{pmatrix}$$
$$\therefore\ \mathcal{O}(\bar{A}, \bar{C}) = \begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix}$$

To find the coordinate transformation T, we use the fact that

$$\mathcal{O}(\bar{A}, \bar{C}) = \mathcal{O}(A, C)T^{-1},$$

which is equivalent to

$$T = \left[\mathcal{O}(\bar{A}, \bar{C}\right]^{-1} \mathcal{O}(A, C).$$

Now, using the formula for the inverse of a 2×2 matrix,

$$\left[\mathcal{O}(\bar{A}, \bar{C})\right]^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} -4 & 1 \\ 1 & 0 \end{pmatrix}$$

Therefore,

$$T = \begin{pmatrix} -4 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 2 & 6 \end{pmatrix} = \begin{pmatrix} -6 & 2 \\ 2 & 1 \end{pmatrix}$$

We will also need the inverse of T later, so let's compute it:

$$T^{-1} = \begin{pmatrix} -6 & 2 \\ 2 & 1 \end{pmatrix}^{-1} = \frac{1}{10} \begin{pmatrix} -1 & 2 \\ 2 & 6 \end{pmatrix}.$$

Now, for the system in OCF with the given $\bar{A}$ and $\bar{C}$, we determine the output injection matrix $\bar{L} = \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix}$, so that the characteristic polynomial of $\bar{A} - \bar{L}\bar{C}$ has a repeated root at -5.

$$\begin{aligned}
&\bar{A} - \bar{L}\bar{C} = \begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix} - \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 - \ell_1 \\ 1 & 4 - \ell_2 \end{pmatrix} \\
&\det(Is - \bar{A} + \bar{L}\bar{C}) = (s+5)^2 \\
&\det \begin{pmatrix} s & \ell_1 - 2 \\ -1 & s + \ell_2 - 4 \end{pmatrix} = (s+5)^2 \\
&s^2 + (\ell_2 - 4)s + \ell_1 - 2 = s^2 + 10s + 25
\end{aligned}$$

Matching coefficients, we get $\ell_1 = 27, \ell_2 = 14$. The observer, in the new coordinates, will have the form

$$\dot{\widehat{\bar{x}}} = (\bar{A} - \bar{L}\bar{C})\widehat{\bar{x}} + \bar{L}y + \bar{B}u,$$

where $\widehat{\bar{x}}$ is the estimate of the state $\bar{x}$. We need to express it in the original coordinates, where $\widehat{x} = T^{-1}\widehat{\bar{x}}$ and $x = T^{-1}\bar{x}$. Thus,

$$\begin{aligned}
\dot{\widehat{x}} &= T^{-1}\dot{\widehat{\bar{x}}} \\
&= T^{-1}\left[\left(TAT^{-1} - \bar{L}CT^{-1}\right)Tx + \bar{L}y + TBu\right] \\
&= (A - T^{-1}\bar{L}C)x + T^{-1}\bar{L}y + Bu \\
&= (A - LC)\widehat{x} + Ly + Bu,
\end{aligned}$$

where $L = T^{-1}\bar{L}$. We can now compute the output injection matrix in the original coordinates (remember that we have already determined the inverse of T):

$$L = \frac{1}{10}\begin{pmatrix} -1 & 2 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} 27 \\ 14 \end{pmatrix} = \begin{pmatrix} 1/10 \\ 138/10 \end{pmatrix}$$

Thus,

$$\begin{aligned} A - LC &= \begin{pmatrix} 0 & 1 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} 1/10 \\ 138/10 \end{pmatrix}\begin{pmatrix} 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} 2/10 & 1/10 \\ 276/10 & 138/10 \end{pmatrix} \\ &= \frac{1}{10}\begin{pmatrix} -2 & 9 \\ -256 & -98 \end{pmatrix} \end{aligned}$$

(which has -5 as a repeated eigenvalue), and the observer dynamics is given by

$$\begin{pmatrix} \dot{\widehat{x}}_1 \\ \dot{\widehat{x}}_2 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} -2 & 9 \\ -256 & -98 \end{pmatrix}\begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{pmatrix} + \frac{1}{10}\begin{pmatrix} 1 \\ 138 \end{pmatrix} y + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u.$$

(d) Design a full-state feedback controller to place controller poles at -1 and -2.

Solution. Since the system is already in CCF, the feedback gain matrix $K = \begin{pmatrix} k_1 & k_2 \end{pmatrix}$ can be determined directly from the fact that

$$\begin{aligned} A - BK &= \begin{pmatrix} 0 & 1 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} k_1 & k_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 2 - k_1 & 4 - k_2 \end{pmatrix}, \end{aligned}$$

and the controller poles are the roots of the characteristic polynomial of $A - BK$. Thus, we want

$$\begin{aligned} \det(Is - A + BK) &= (s+1)(s+2) \\ \det\begin{pmatrix} s & -1 \\ k_1 - 2 & s + k_2 - 4 \end{pmatrix} &= s^2 + 3s + 2 \\ s^2 + (k_2 - 4)s + k_1 - 2 &= s^2 + 3s + 2. \end{aligned}$$

Matching coefficients, we get $k_1 = 4$ and $k_2 = 7$. Thus, the controller will be

$$u = -K\widehat{x} = -\begin{pmatrix} 4 & 7 \end{pmatrix}\begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{pmatrix}$$

(e) Suppose that you now apply the controller you designed in part (d) to the state estimate $\hat{x}$ produced by the observer you designed in part (c) – that is, $u = -K\hat{x}$. Write down the transfer function of the resulting *dynamic output feedback controller* (observer+controller) from Y to U.

Solution. The observer-controller dynamics has the form

$$\dot{\hat{x}} = (A - LC - BK)\hat{x} + Ly$$
$$u = -K\hat{x}.$$

The transfer function from Y to U is therefore given by

$$U(s) = -K(Is - A + LC + BK)^{-1}LY(s)$$

From our solution above,

$$\begin{aligned}
K &= \begin{pmatrix} 4 & 7 \end{pmatrix} \\
A - LC - BK &= \frac{1}{10}\begin{pmatrix} -2 & 9 \\ -256 & -98 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 4 & 7 \end{pmatrix} \\
&= \frac{1}{10}\left[\begin{pmatrix} -2 & 9 \\ -256 & -98 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 40 & 70 \end{pmatrix}\right] \\
&= \frac{1}{10}\begin{pmatrix} -2 & 9 \\ -296 & -168 \end{pmatrix} \\
(Is - A + LC + BK)^{-1} &= \begin{pmatrix} s + \frac{2}{10} & -\frac{9}{10} \\ \frac{296}{10} & s + \frac{168}{10} \end{pmatrix}^{-1} \\
&= \frac{1}{s^2 + 17s + 30}\begin{pmatrix} s + \frac{168}{10} & \frac{9}{10} \\ -\frac{296}{10} & s + \frac{2}{10} \end{pmatrix} \\
(Is - A + LC + BK)^{-1}L &= \frac{1}{10s^2 + 170s + 300}\begin{pmatrix} s + \frac{168}{10} & \frac{9}{10} \\ -\frac{296}{10} & s + \frac{2}{10} \end{pmatrix}\begin{pmatrix} 1 \\ 138 \end{pmatrix} \\
&= \frac{1}{10s^2 + 170s + 300}\begin{pmatrix} s + 141 \\ 138s + 2 \end{pmatrix} \\
-K(Is - A + LC + BK)^{-1}L &= -\frac{1}{10s^2 + 170s + 300}\begin{pmatrix} 4 & 7 \end{pmatrix}\begin{pmatrix} s + 141 \\ 138s + 2 \end{pmatrix} \\
&= -\frac{970s + 578}{10s^2 + 170s + 300}
\end{aligned}$$

Therefore, since our system is SISO, the transfer function is

$$\frac{U(s)}{Y(s)} = -\frac{970s + 578}{10s^2 + 170s + 300}$$