Plan of the Lecture

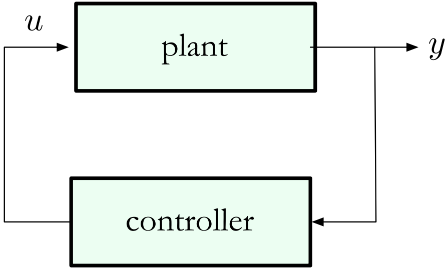
- Review: observability; Luenberger observer and state estimation error.
- Today's topic: joint observer and controller design: dynamic output feedback.

Goal: learn how to design an observer and a controller to achieve accurate closed-loop pole placement.

Reading: FPE, Chapter 7

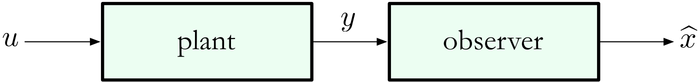
Is Full State Feedback Always Available?

In a typical system, measurements are provided by sensors:



Full state feedback $u = -Kx$ is *not implementable*!!

In that case, an observer is used to estimate the state x:

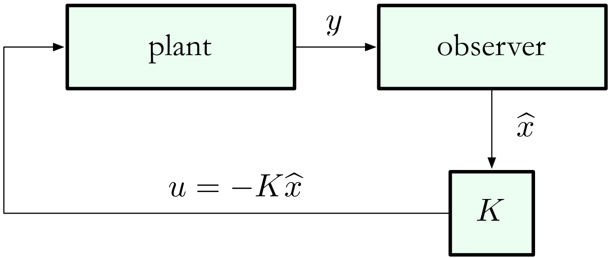


State Estimation Using an Observer

If the system is observable, the state estimate $\widehat{x}$ is *asymptotically accurate*:

$$\|\widehat{x}(t) - x(t)\| = \sqrt{\sum_{i=1}^{n} \left(\widehat{x}_i(t) - x_i(t)\right)^2} \xrightarrow{t \to \infty} 0$$

If we are successful, then we can try estimated state feedback:



Observability

Consider a single-output system ($y \in \mathbb{R}$):

$$\dot{x} = Ax + Bu, \qquad y = Cx \qquad\qquad x \in \mathbb{R}^n$$

The Observability Matrix is defined as

$$\mathcal{O}(A, C) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

We say that the above system is observable if its observability matrix $\mathcal{O}(A, C)$ is *invertible*.

(This definition is only true for the single-output case; the multiple-output case involves the *rank* of $\mathcal{O}(A, C)$.)

Observer Canonical Form

A single-output state-space model

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

is said to be in Observer Canonical Form (OCF) if the matrices A, C are of the form

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & * \\ 1 & 0 & \dots & 0 & 0 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & * \\ 0 & 0 & \dots & 0 & 1 & * \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Fact: A system in OCF is *always observable*!!

(The proof of this for $n > 2$ uses the Jordan canonical form, we will not worry about this.)

The Luenberger Observer

$$\begin{aligned} \text{System:} \qquad & \dot{x} = Ax \\ & y = Cx \\ \text{Observer:} \qquad & \dot{\widehat{x}} = (A - LC)\widehat{x} + Ly. \end{aligned}$$

What happens to state estimation error $e = x - \widehat{x}$ as $t \to \infty$?

$$\dot{e} = (A - LC)e$$

Does $e(t)$ converge to zero in some sense?

The Luenberger Observer

$$\begin{aligned} \text{System:} \qquad & \dot{x} = Ax \\ & y = Cx \\ \text{Observer:} \qquad & \dot{\widehat{x}} = (A - LC)\widehat{x} + Ly \\ \text{Error:} \qquad & \dot{e} = (A - LC)e \end{aligned}$$

Recall our assumption that $A - LC$ is Hurwitz (all eigenvalues are in LHP). This implies that

$$\|x(t) - \widehat{x}(t)\|^2 = \|e(t)\|^2 = \sum_{i=1}^{n} |e_i(t)|^2 \xrightarrow{t\to\infty} 0$$

at an exponential rate, determined by the eigenvalues of $A - LC$.

For fast convergence, want eigenvalues of $A - LC$ far into LHP!!

Observability and Estimation Error

Fact: If the system

$$\dot{x} = Ax, \qquad y = Cx$$

is observable, then we can arbitrarily assign eigenvalues of $A - LC$ by a suitable choice of the output injection matrix L.

This is similar to the fact that controllability implies arbitrary closed-loop pole placement by state feedback.

In fact, these two facts are closely related because CCF is dual to OCF.

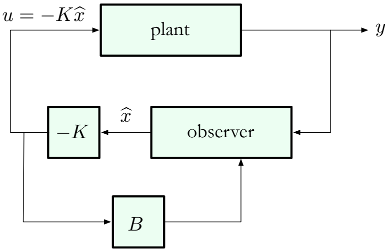
Combining Full-State Feedback with an Observer

- So far, we have focused on autonomous systems ($u = 0$).
- What about nonzero inputs?

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

— assume (A, B) is controllable and (A, C) is observable.

- Today, we will learn how to use an observer together with estimated state feedback to (approximately) place closed-loop poles.



Combining Full-State Feedback with an Observer

- Consider

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

  where (A, B) is controllable and (A, C) is observable.
- We know how to find K, such that $A - BK$ has desired eigenvalues (controller poles).
- Since we do not have access to x, we must design an observer. But this time, we need a slight modification because of the Bu term.

Observer in the Presence of Control Input

- Let's see what goes wrong when we use the old approach:

$$\dot{\widehat{x}} = (A - LC)\widehat{x} + Ly$$

- For the estimation error $e = x - \widehat{x}$, we have

$$\begin{aligned}\dot{e} &= \dot{x} - \dot{\widehat{x}} \\ &= Ax + Bu - [(A - LC)\widehat{x} + LCx] \\ &= (A - LC)e + Bu \qquad \text{– not good}\end{aligned}$$

- Idea: since u is a signal we can access, let's use it as an input to the observer to cancel the Bu term from $\dot{x}$.
- Modified observer:

$$\begin{aligned}\dot{\widehat{x}} &= (A - LC)\widehat{x} + Ly + Bu \\ \dot{e} &= \dot{x} - \dot{\widehat{x}} \\ &= Ax + Bu - [(A - LC)\widehat{x} + LCx + Bu] \\ &= (A - LC)e \qquad \text{regardless of } u\end{aligned}$$

Observer and Controller

$$
\begin{aligned}
\text{System:} \quad & \dot{x} = Ax + Bu \\
& y = Cx \\
\text{Observer:} \quad & \dot{\widehat{x}} = (A - LC)\widehat{x} + Ly + Bu \\
\text{Error:} \quad & \dot{e} = (A - LC)e
\end{aligned}
$$

- By observability, we can arbitrarily assign $\text{eig}(A - LC)$; these should be farther into LHP than desired controller poles.

$$
\text{Controller:} \quad u = -K\widehat{x} \quad (\text{estimated state feedback})
$$

- By controllability, we can arbitrarily assign $\text{eig}(A - BK)$.

Observer and Controller

$$\begin{aligned} \text{System:} \quad & \dot{x} = Ax + Bu \\ & y = Cx \\ \text{Observer:} \quad & \dot{\widehat{x}} = (A - LC)\widehat{x} + Ly + Bu \\ \text{Controller:} \quad & u = -K\widehat{x} \end{aligned}$$

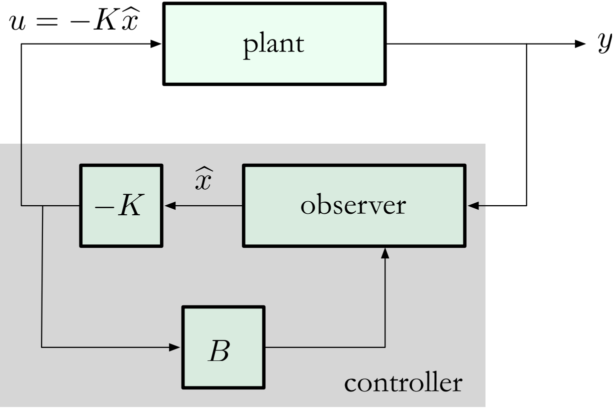
The overall observer-controller system is:

$$\begin{aligned} \dot{\widehat{x}} &= (A - LC)\widehat{x} + Ly + B\underbrace{(-K\widehat{x})}_{=u} \\ &= (A - LC - BK)\widehat{x} + Ly \\ u &= -K\widehat{x} \qquad \text{(dynamic output feedback)} \end{aligned}$$

— this is a dynamical system with input y and output u

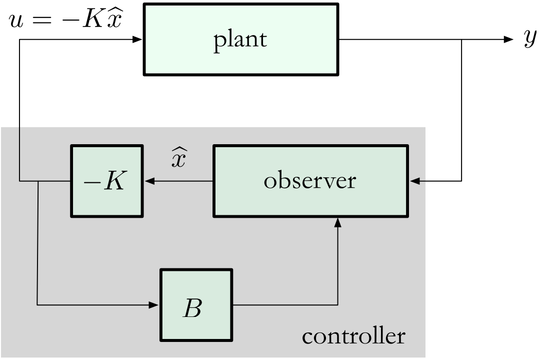
Dynamic Output Feedback

$$\dot{x} = Ax + Bu$$
$$y = Cx$$
$$\dot{\widehat{x}} = (A - LC - BK)\widehat{x} + Ly$$
$$u = -K\widehat{x}$$



Dynamic Output Feedback

$$\dot{\hat{x}} = (A - LC - BK)\hat{x} + Ly, \quad u = -K\hat{x}$$



Controller transfer function (from y to u):

$$s\hat{X} = (A - LC - BK)\hat{X} + LY, \quad U = -K\hat{X}$$

$$U = \underbrace{-K(Is - A + LC + BK)^{-1}L}_{=D(s)}Y$$

Dynamic Output Feedback: Does It Work?

Summarizing:

- When $y = x$, full state feedback $u = -Kx$ achieves desired pole placement.
- How do we know that $u = -K\widehat{x}$ achieves similar objectives?

Here is our overall closed-loop system:

$$\begin{aligned} \dot{x} &= Ax - BK\widehat{x} \\ \dot{\widehat{x}} &= (A - LC - BK)\widehat{x} + LCx \end{aligned}$$

We can write it in block matrix form:

$$\begin{pmatrix} \dot{x} \\ \dot{\widehat{x}} \end{pmatrix} = \begin{pmatrix} A & -BK \\ LC & A - LC - BK \end{pmatrix} \begin{pmatrix} x \\ \widehat{x} \end{pmatrix}$$

How do we relate this to “nominal” behavior, $A - BK$?

Dynamic Output Feedback

$$\begin{pmatrix} \dot{x} \\ \dot{\widehat{x}} \end{pmatrix} = \begin{pmatrix} A & -BK \\ LC & A - LC - BK \end{pmatrix} \begin{pmatrix} x \\ \widehat{x} \end{pmatrix}$$

Let us transform to new coordinates:

$$\begin{pmatrix} x \\ \widehat{x} \end{pmatrix} \longmapsto \begin{pmatrix} x \\ e \end{pmatrix} = \begin{pmatrix} x \\ x - \widehat{x} \end{pmatrix} = \underbrace{\begin{pmatrix} I & 0 \\ I & -I \end{pmatrix}}_{T} \begin{pmatrix} x \\ \widehat{x} \end{pmatrix}$$

Two key observations:

- T is invertible, so the new representation is equivalent to the old one
- in the new coordinates, we have

$$\begin{aligned} \dot{x} &= Ax - BK\widehat{x} \\ &= (A - BK)x + BK(x - \widehat{x}) \\ &= (A - BK)x + BKe \\ \dot{e} &= (A - LC)e \end{aligned}$$

The Main Result: Separation Principle

So now we can write

$$\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \underbrace{\begin{pmatrix} A - BK & BK \\ 0 & A - LC \end{pmatrix}}_{\text{upper triangular matrix}} \begin{pmatrix} x \\ e \end{pmatrix}$$

The closed-loop characteristic polynomial is

$$\det \begin{pmatrix} Is - A + BK & -BK \\ 0 & Is - A + LC \end{pmatrix}$$
$$= \det (Is - A + BK) \cdot \det (Is - A + LC)$$

Separation principle. The closed-loop eigenvalues are:

$$\{\text{controller poles (roots of } \det(Is - A + BK))\}$$
$$\cup \{\text{observer poles (roots of } \det(Is - A + LC))\}$$

— this holds only for linear systems!!

Separation Principle

Separation principle. The closed-loop eigenvalues are:

$$\begin{aligned}&\{\text{controller poles (roots of } \det(Is - A + BK))\}\\&\quad\cup\{\text{observer poles (roots of } \det(Is - A + LC))\}\end{aligned}$$

— this holds only for linear systems!!

Moral of the story:

- If we choose observer poles to be several times faster than the controller poles (e.g., 2–5 times), then the controller poles will be dominant.
- Dynamic output feedback gives essentially the same performance as (nonimplementable) full-state feedback — provided observer poles are far enough into LHP.
- Remember: the system must be controllable and observable!!