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Note!! The way I teach the Root Locus differs a bit from what the textbook does (good news: it is simpler). Still, pay attention in class!!

Course structure so far:

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$\downarrow$		
analysis	—	transfer function, response, stability
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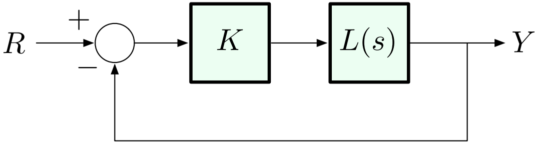
We will focus on design from now on.

The Root Locus Design Method

(invented by Walter R. Evans in 1948)

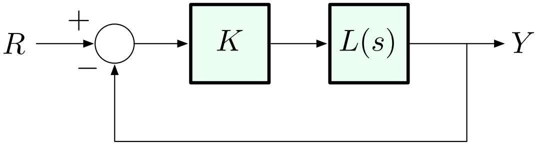
Consider this unity feedback configuration:



where

- K is a constant gain
- $L(s) = \dfrac{b(s)}{a(s)}$, where $a(s)$ and $b(s)$ are some polynomials

The Root Locus Design Method



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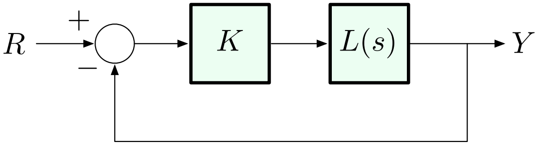

Closed-loop transfer function:

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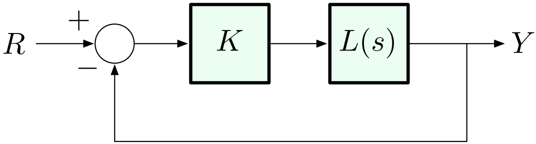

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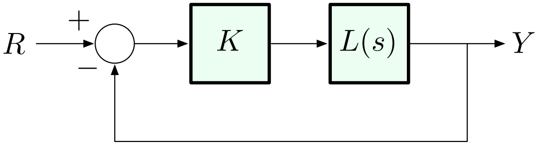

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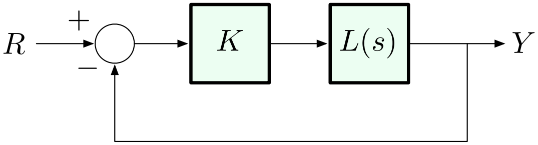

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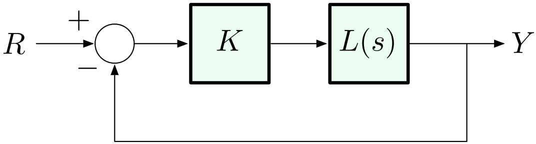

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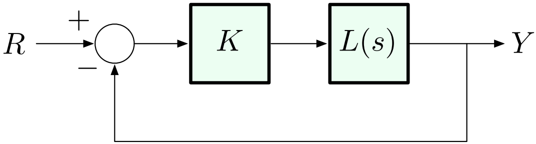

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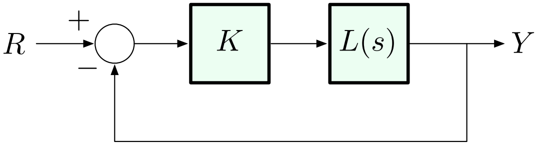

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$$\underbrace{a(s) + Kb(s)}_{\substack{\text{characteristic}\\ \text{polynomial}}} = 0 \qquad \text{characteristic equation}$$

A Comment on Change of Notation

Note the change of notation:

$$\text{from } H(s) \text{ or } G(s) = \frac{q(s)}{p(s)} \quad \text{to } L(s) = \frac{b(s)}{a(s)}$$

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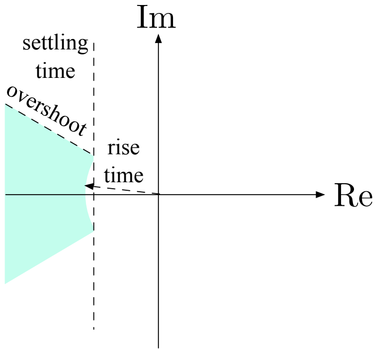
As long as we can represent the poles of the closed-loop transfer function as roots of the equation $1 + KL(s) = 0$ for *some choice* of K and $L(s)$, we can apply the RL method.

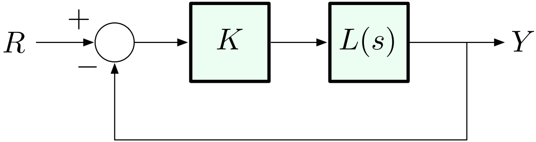
Towards Quantitative Characterization of Stability

Qualitative description of stability: Routh test gives us a range of K to guarantee stability.



For what values of K do we best satisfy given design specs?

Root Locus and Quantitative Stability



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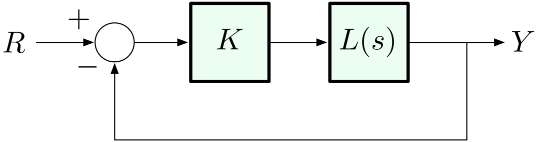

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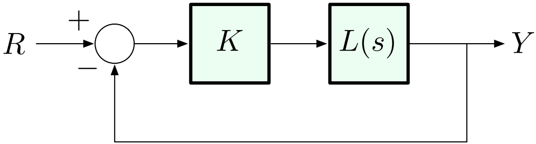

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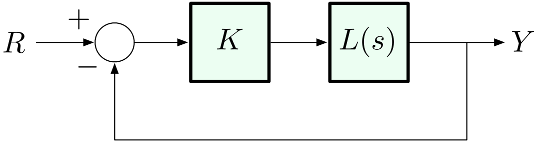

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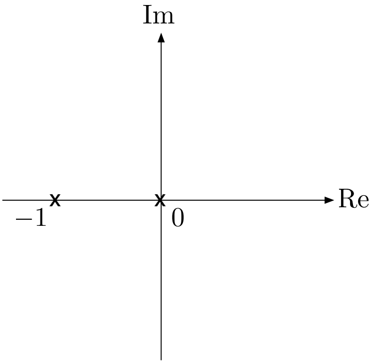

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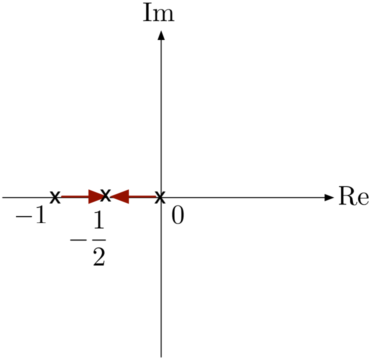

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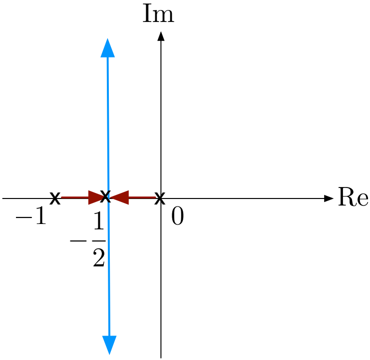

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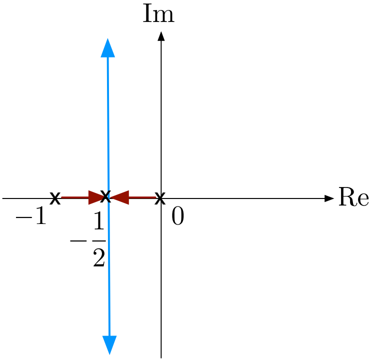

($s = -1/2$ is the *point of breakaway* from the real axis)

Example, continued

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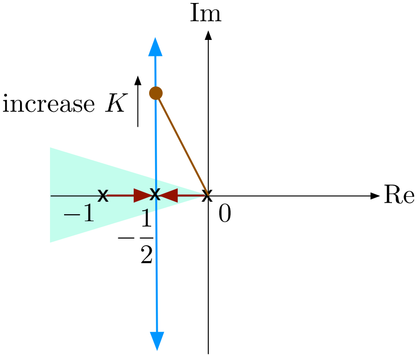

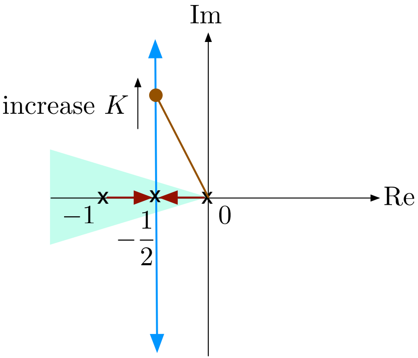

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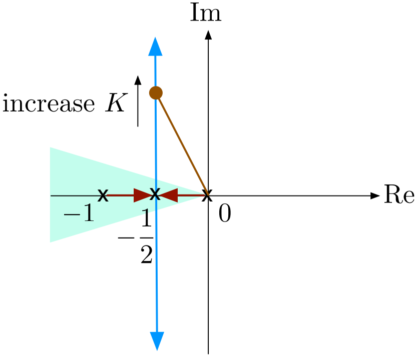

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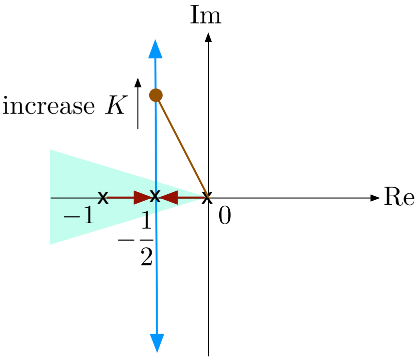

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Our goal: develop simple rules for (approximately) sketching the root locus in the general case.

Equivalent Characterization of RL: Phase Condition

Recall our original definition: The *root locus* for $1 + KL(s)$ is the set of all closed-loop poles, i.e., the roots of

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This gives us an equivalent characterization:

The phase condition: The root locus of $1 + KL(s)$ is the set of all $s \in \mathbb{C}$, such that $\angle L(s) = 180^\circ$, i.e., $L(s)$ is real and negative.

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- Rule E — asymptotes
- Rule F — $j\omega$-crossings

Today, we will cover mostly Rules A–C (and a bit of D).

Rule A: Number of Branches

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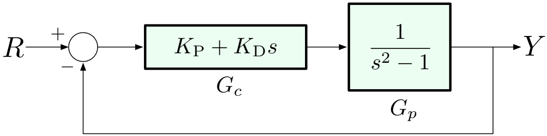
Note: if $n > m$, we have n branches, but only m zeros. The remaining $n - m$ branches go off to infinity (end at "zeros at infinity").

Example

PD control of an unstable 2nd-order plant



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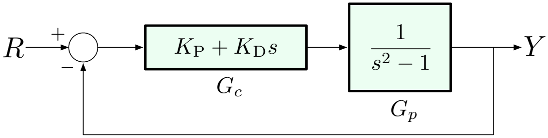

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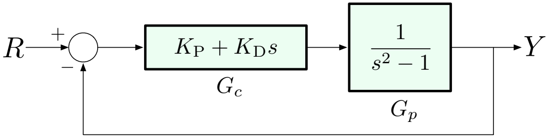

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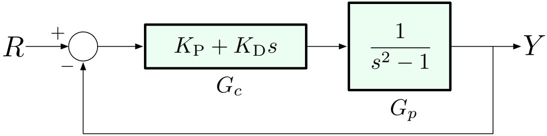

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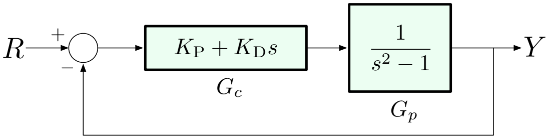

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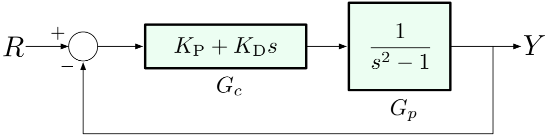

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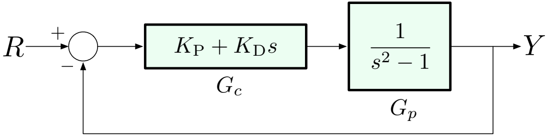

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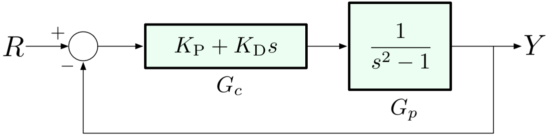

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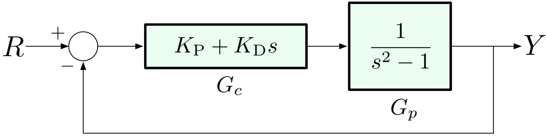

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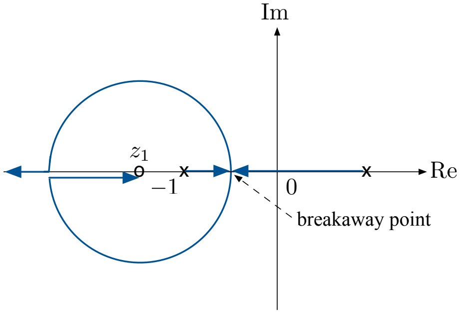
So the root locus will look something like this:



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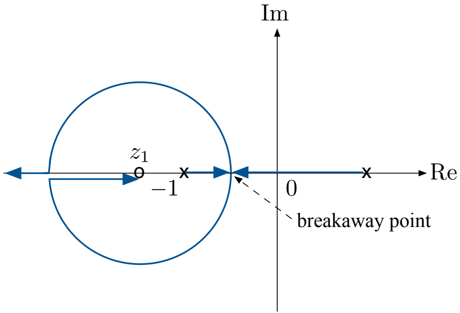

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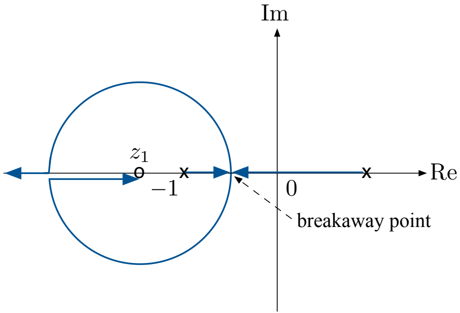

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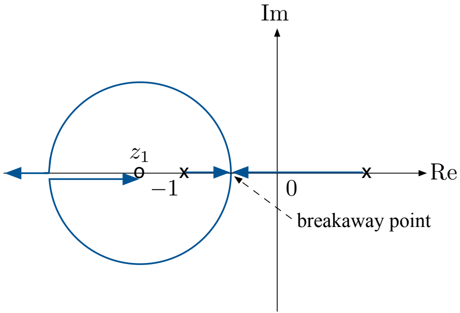

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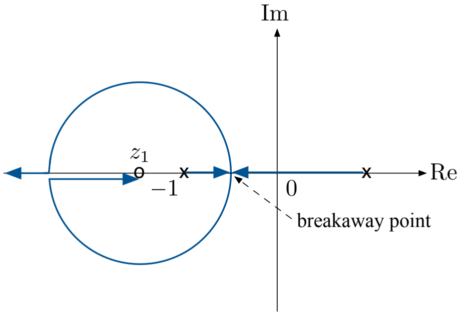

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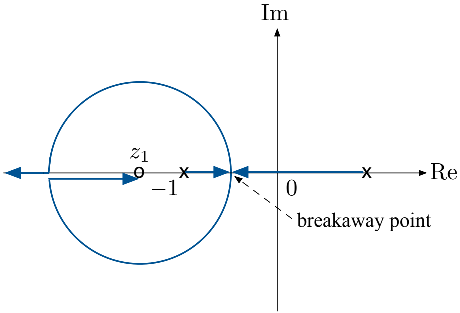

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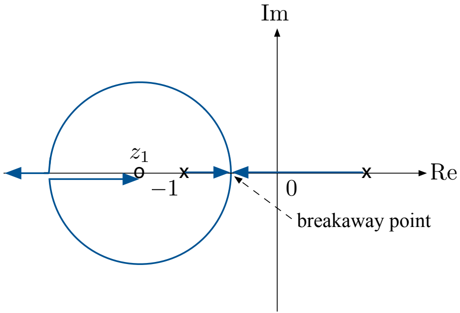

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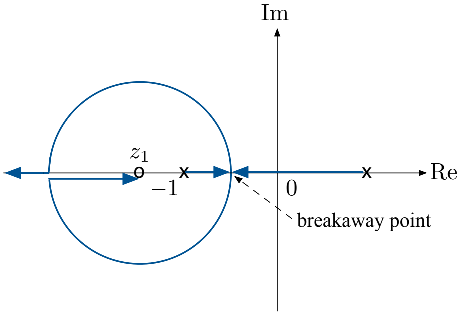

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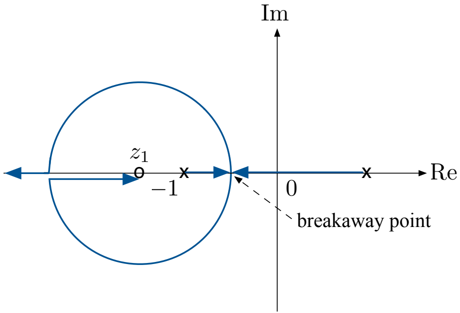

Is the point $s = 0$ on the root locus?

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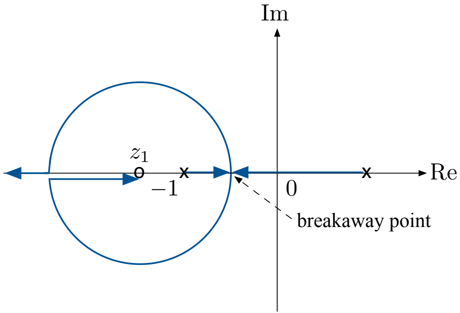

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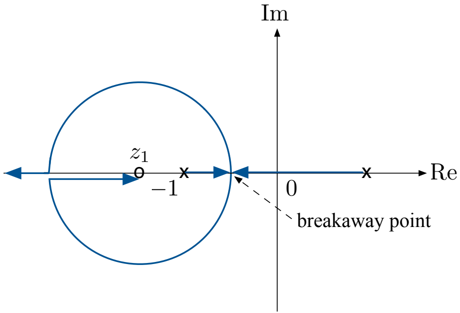

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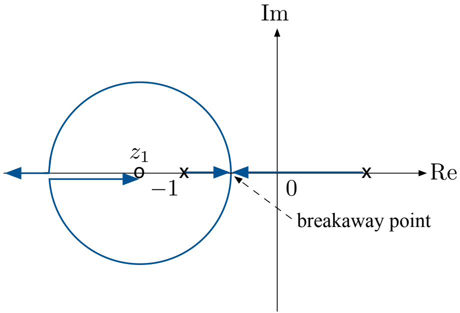

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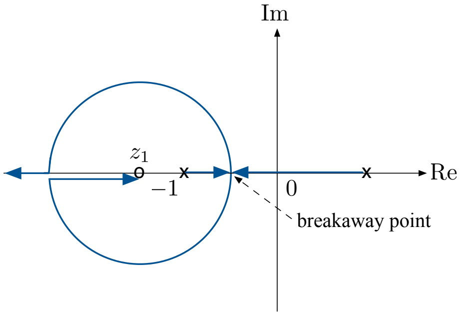

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For concreteness, let's see what happens when

$$K_{\mathrm{P}}/K_{\mathrm{D}} = -z_1 = 2 \qquad \text{and} \qquad K = K_{\mathrm{D}} = 5 \Longrightarrow K_{\mathrm{D}} = 10$$

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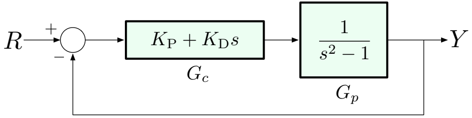

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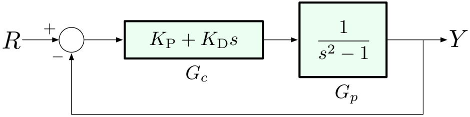

$$G_c(s) = 10 + 5s$$

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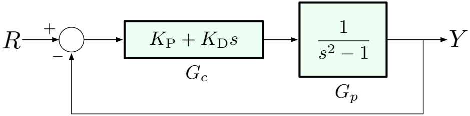

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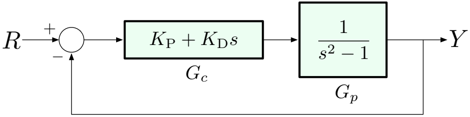

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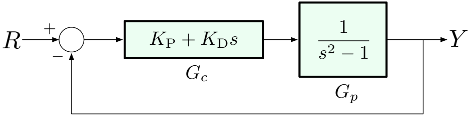

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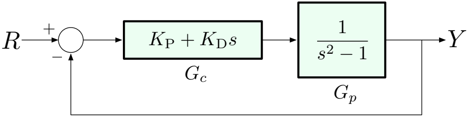
$$s^2 + 5s + 9 = 0$$

From Root Locus to Time Response Specs

For concreteness, let's see what happens when

$$K_\mathrm{P}/K_\mathrm{D} = -z_1 = 2 \qquad \text{and} \qquad K = K_\mathrm{D} = 5 \Longrightarrow K_\mathrm{D} = 10$$



$$G_c(s) = 10 + 5s$$

$$u = 10e + 5\dot{e}, \qquad e = r - y$$

$$\text{Characteristic equation:} \quad 1 + 5\left(\frac{s+2}{s^2-1}\right) = 0$$

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Relate to 2nd-order response: $\omega_n^2 = 9,\ 2\zeta\omega_n = 5 \Longrightarrow \zeta = 5/6$

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Rules D–F!!

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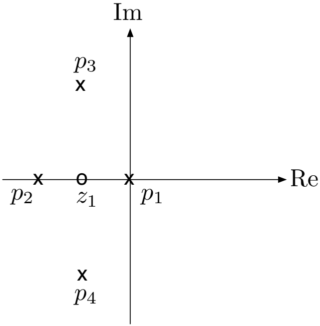

Example, continued

Three more rules:

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- Rule D: real locus
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Rules D and E are both based on the fact that

$$1 + KL(s) = 0 \text{ for some } K > 0 \quad \Longleftrightarrow \quad L(s) < 0$$

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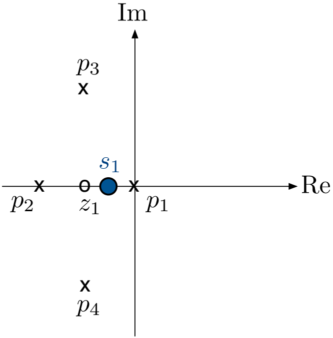
\end{aligned}$$

— this sum must be $\pm 180^\circ$ for *any* s that lies on the RL.

Rule D: Real Locus

So, we try test points:



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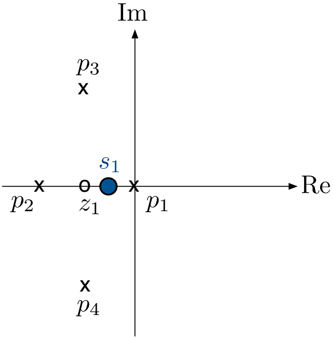

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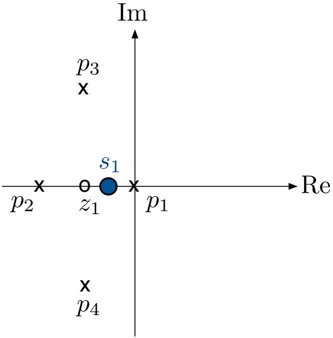

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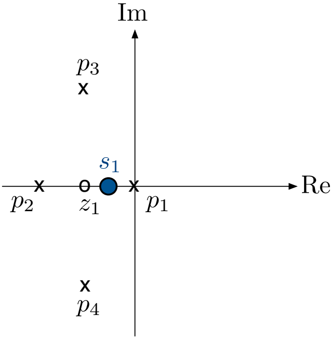

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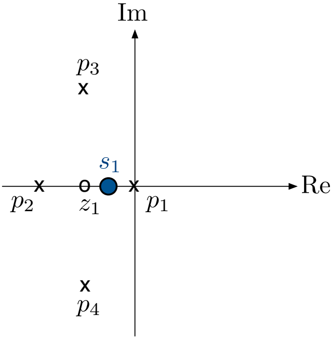

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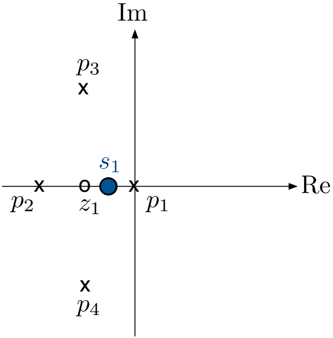

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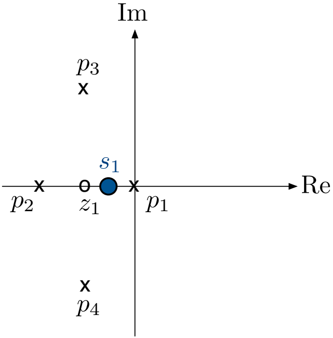

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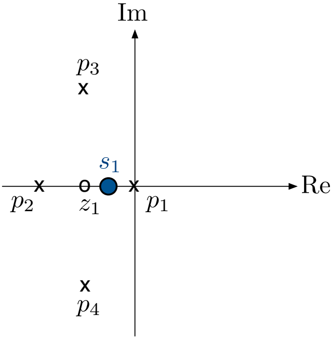

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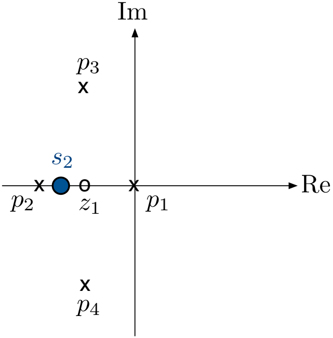

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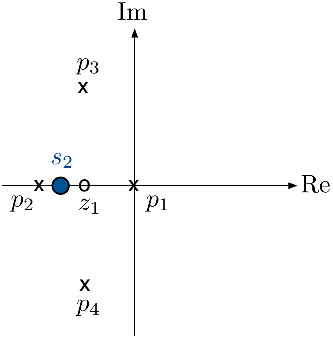

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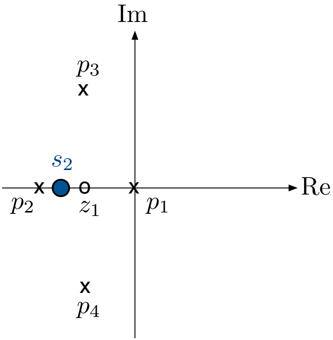

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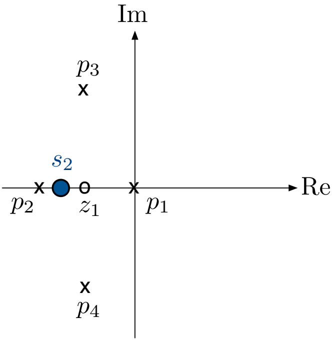

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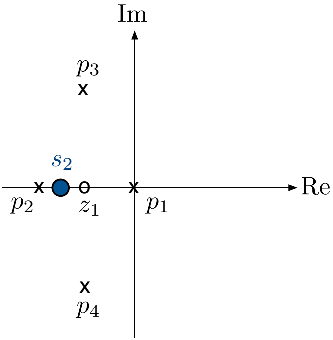

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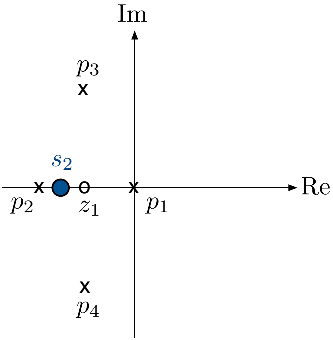

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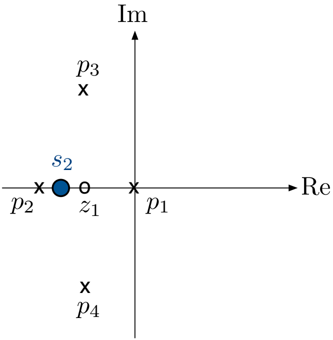

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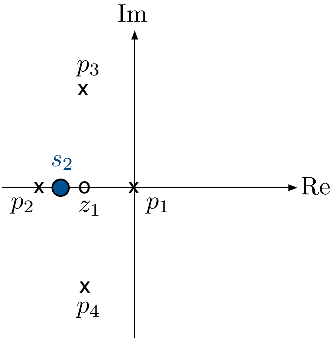

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$\times$ s_1 is not on RL

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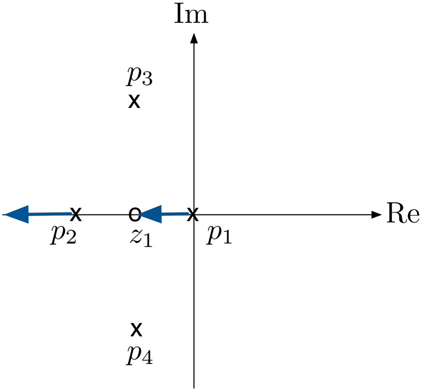

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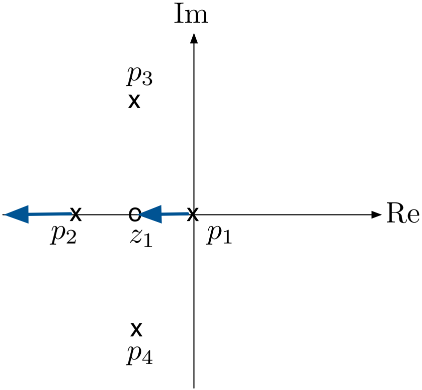

We will cover Rules E and F, and complete the RL for this example, in the next lecture.