Plan of the Lecture

- Review: prototype 2nd-order system
- Today's topic: transient response specifications

Plan of the Lecture

- Review: prototype 2nd-order system
- Today's topic: transient response specifications

Goal: develop formulas and intuition for various features of the transient response: rise time, overshoot, settling time.

Plan of the Lecture

- Review: prototype 2nd-order system
- Today's topic: transient response specifications

Goal: develop formulas and intuition for various features of the transient response: rise time, overshoot, settling time.

Reading: FPE, Sections 3.3–3.4; lab manual

Prototype 2nd-Order System

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

By the quadratic formula, the poles are:

$$\begin{aligned} s &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \\ &= -\omega_n\left(\zeta \pm \sqrt{\zeta^2 - 1}\right) \end{aligned}$$

The nature of the poles changes depending on ζ:

- $\zeta > 1$ both poles are real and negative
- $\zeta = 1$ one negative pole
- $\zeta < 1$ two complex poles with negative real parts

$$s = -\sigma \pm j\omega_d$$

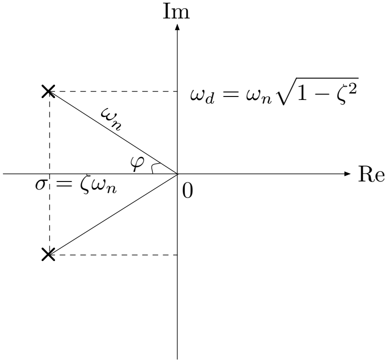
$$\text{where} \qquad \sigma = \zeta\omega_n, \ \omega_d = \omega_n\sqrt{1 - \zeta^2}$$

Prototype 2nd-Order System

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \qquad \zeta < 1$$

The poles are

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\sigma \pm j\omega_d$$



Note that

$$\begin{aligned}\sigma^2 + \omega_d^2 &= \zeta^2\omega_n^2 + \omega_n^2 - \zeta^2\omega_n^2 \\ &= \omega_n^2 \\ \cos\varphi &= \frac{\zeta\omega_n}{\omega_n} = \zeta\end{aligned}$$

2nd-Order Response

Let's compute the system's impulse and step response:

2nd-Order Response

Let's compute the system's impulse and step response:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

2nd-Order Response

Let's compute the system's impulse and step response:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

- Impulse response:

$$h(t) = \mathscr{L}^{-1}\{H(s)\} = \mathscr{L}^{-1}\left\{\frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}\right\}$$

2nd-Order Response

Let's compute the system's impulse and step response:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

- Impulse response:

$$h(t) = \mathscr{L}^{-1}\{H(s)\} = \mathscr{L}^{-1}\left\{\frac{(\omega_n^2/\omega_d)\omega_d}{(s+\sigma)^2 + \omega_d^2}\right\}$$

2nd-Order Response

Let's compute the system's impulse and step response:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

- Impulse response:

$$\begin{aligned} h(t) &= \mathscr{L}^{-1}\{H(s)\} = \mathscr{L}^{-1}\left\{\frac{(\omega_n^2/\omega_d)\omega_d}{(s+\sigma)^2 + \omega_d^2}\right\} \\ &= \frac{\omega_n^2}{\omega_d} e^{-\sigma t} \sin(\omega_d t) \end{aligned}$$

2nd-Order Response

Let's compute the system's impulse and step response:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

- Impulse response:

$$\begin{aligned} h(t) &= \mathscr{L}^{-1}\{H(s)\} = \mathscr{L}^{-1}\left\{\frac{(\omega_n^2/\omega_d)\omega_d}{(s+\sigma)^2 + \omega_d^2}\right\} \\ &= \frac{\omega_n^2}{\omega_d} e^{-\sigma t} \sin(\omega_d t) \qquad \text{(table, \# 20)} \end{aligned}$$

2nd-Order Response

Let's compute the system's impulse and step response:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

- Impulse response:

$$\begin{aligned} h(t) &= \mathscr{L}^{-1}\{H(s)\} = \mathscr{L}^{-1}\left\{\frac{(\omega_n^2/\omega_d)\omega_d}{(s+\sigma)^2 + \omega_d^2}\right\} \\ &= \frac{\omega_n^2}{\omega_d} e^{-\sigma t}\sin(\omega_d t) \qquad \text{(table, \# 20)} \end{aligned}$$

- Step response:

$$\mathscr{L}^{-1}\left\{\frac{H(s)}{s}\right\} = \mathscr{L}^{-1}\left\{\frac{\omega_n^2}{s[(s+\sigma)^2 + \omega_d^2]}\right\}$$

2nd-Order Response

Let's compute the system's impulse and step response:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

- Impulse response:

$$h(t) = \mathscr{L}^{-1}\{H(s)\} = \mathscr{L}^{-1}\left\{\frac{(\omega_n^2/\omega_d)\omega_d}{(s+\sigma)^2 + \omega_d^2}\right\}$$

$$= \frac{\omega_n^2}{\omega_d} e^{-\sigma t} \sin(\omega_d t) \qquad \text{(table, \# 20)}$$

- Step response:

$$\mathscr{L}^{-1}\left\{\frac{H(s)}{s}\right\} = \mathscr{L}^{-1}\left\{\frac{\sigma^2 + \omega_d^2}{s[(s+\sigma)^2 + \omega_d^2]}\right\}$$

2nd-Order Response

Let's compute the system's impulse and step response:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

- Impulse response:

$$\begin{aligned} h(t) = \mathscr{L}^{-1}\{H(s)\} &= \mathscr{L}^{-1}\left\{\frac{(\omega_n^2/\omega_d)\omega_d}{(s+\sigma)^2 + \omega_d^2}\right\} \\ &= \frac{\omega_n^2}{\omega_d} e^{-\sigma t}\sin(\omega_d t) \qquad \text{(table, \# 20)} \end{aligned}$$

- Step response:

$$\mathscr{L}^{-1}\left\{\frac{H(s)}{s}\right\} = \mathscr{L}^{-1}\left\{\frac{\sigma^2 + \omega_d^2}{s[(s+\sigma)^2 + \omega_d^2]}\right\}$$

$$= 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

2nd-Order Response

Let's compute the system's impulse and step response:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

- Impulse response:

$$h(t) = \mathscr{L}^{-1}\{H(s)\} = \mathscr{L}^{-1}\left\{\frac{(\omega_n^2/\omega_d)\omega_d}{(s+\sigma)^2 + \omega_d^2}\right\}$$

$$= \frac{\omega_n^2}{\omega_d} e^{-\sigma t} \sin(\omega_d t) \qquad \text{(table, \# 20)}$$

- Step response:

$$\mathscr{L}^{-1}\left\{\frac{H(s)}{s}\right\} = \mathscr{L}^{-1}\left\{\frac{\sigma^2 + \omega_d^2}{s[(s+\sigma)^2 + \omega_d^2]}\right\}$$

$$= 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right) \qquad \text{(table, \#21)}$$

2nd-Order Step Response

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

$$u(t) = 1(t) \qquad \longrightarrow \qquad y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

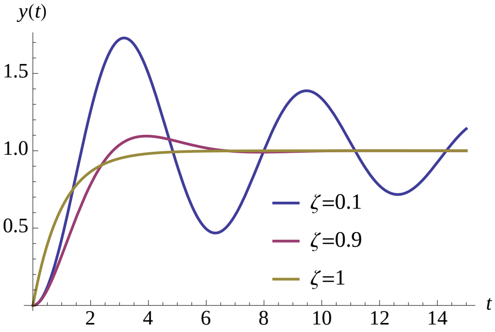
where $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$ (damped frequency)

2nd-Order Step Response

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

$$u(t) = 1(t) \qquad \longrightarrow \qquad y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

where $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$ (damped frequency)



The parameter ζ is called the *damping ratio*

- $\zeta > 1$: system is overdamped
- $\zeta < 1$: system is underdamped
- $\zeta = 0$: no damping ($\omega_d = \omega_n$)

2nd-Order Step Response

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

$$u(t) = 1(t) \qquad \longrightarrow \qquad y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

where $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$ (damped frequency)

2nd-Order Step Response

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

$$u(t) = 1(t) \qquad \longrightarrow \qquad y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

where $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$ (damped frequency)

We will see that the parameters ζ and ω_n determine certain important features of the transient part of the above step response.

2nd-Order Step Response

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

$$u(t) = 1(t) \qquad \longrightarrow \qquad y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

where $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$ (damped frequency)

We will see that the parameters ζ and ω_n determine certain important features of the transient part of the above step response.

We will also learn how to pick ζ and ω_n in order to *shape* these features according to given *specifications*.

Transient Response Specifications: Rise Time

Let's first take a look at *1st-order step response*

Transient Response Specifications: Rise Time

Let's first take a look at *1st-order step response*

$$H(s) = \frac{a}{s+a}, \qquad a > 0$$

Transient Response Specifications: Rise Time

Let's first take a look at *1st-order step response*

$$H(s) = \frac{a}{s+a}, \qquad a > 0 \qquad \text{(stable pole)}$$

Transient Response Specifications: Rise Time

Let's first take a look at *1st-order step response*

$$H(s) = \frac{a}{s+a}, \qquad a > 0 \qquad \text{(stable pole)}$$

DC gain = 1 (by FVT)

Transient Response Specifications: Rise Time

Let's first take a look at *1st-order step response*

$$H(s) = \frac{a}{s+a}, \qquad a > 0 \qquad \text{(stable pole)}$$

DC gain = 1 (by FVT)

Step response:

Transient Response Specifications: Rise Time

Let's first take a look at *1st-order step response*

$$H(s) = \frac{a}{s+a}, \qquad a > 0 \qquad \text{(stable pole)}$$

DC gain $= 1$ (by FVT)

Step response: $$Y(s) = \frac{H(s)}{s} = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a}$$

Transient Response Specifications: Rise Time

Let's first take a look at *1st-order step response*

$$H(s) = \frac{a}{s+a}, \qquad a > 0 \qquad \text{(stable pole)}$$

DC gain = 1 (by FVT)

Step response: $$Y(s) = \frac{H(s)}{s} = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a}$$

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = 1(t) - e^{-at}$$

Transient Response Specifications: Rise Time

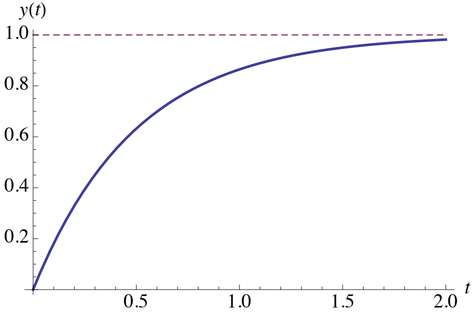
Let's first take a look at *1st-order step response*

$$H(s) = \frac{a}{s+a}, \qquad a > 0 \qquad \text{(stable pole)}$$

DC gain = 1 (by FVT)

Step response: $$Y(s) = \frac{H(s)}{s} = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a}$$

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = 1(t) - e^{-at}$$



Transient Response Specifications: Rise Time

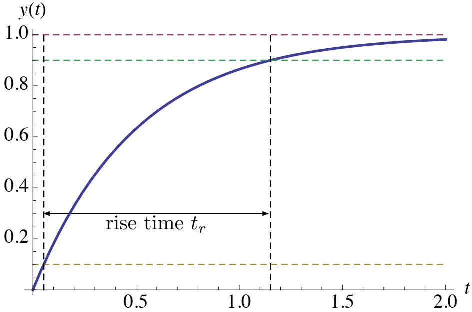
Let's first take a look at *1st-order step response*

$$H(s) = \frac{a}{s+a}, \qquad a > 0 \qquad \text{(stable pole)}$$

DC gain $= 1$ (by FVT)

Step response: $$Y(s) = \frac{H(s)}{s} = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a}$$

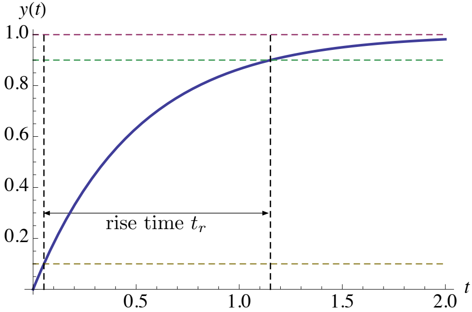
$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = 1(t) - e^{-at}$$



Rise time t_r: the time it takes to get from 10% of steady-state value to 90%

Rise Time

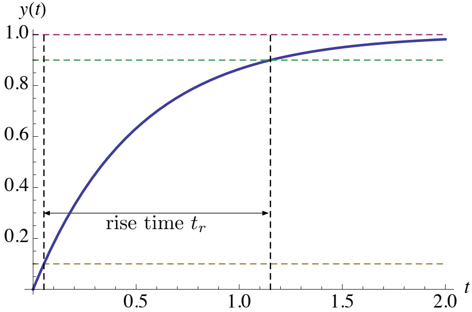
Step response: $y(t) = 1(t) - e^{-at}$



Rise time t_r: the time it takes to get from 10% of steady-state value to 90%

Rise Time

Step response: $y(t) = 1(t) - e^{-at}$



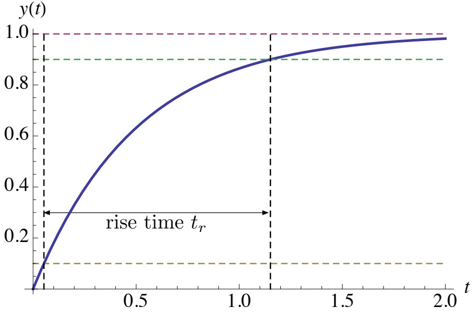
Rise time t_r: the time it takes to get from 10% of steady-state value to 90%

Examples of rise time:

- car — going from 0 to 60 mph in 7 sec
- oven — reach desired preheat temperature quickly
- thermostat, building climate control
- other examples?

Rise Time

Step response: $y(t) = 1(t) - e^{-at}$

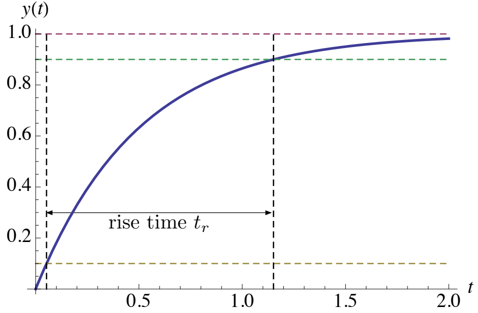


Rise time t_r: the time it takes to get from 10% of steady-state value to 90%

In this example, it is easy to compute t_r analytically:

Rise Time

Step response: $y(t) = 1(t) - e^{-at}$



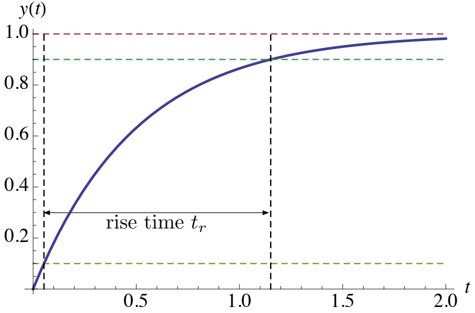
Rise time t_r: the time it takes to get from 10% of steady-state value to 90%

In this example, it is easy to compute t_r analytically:

$$1 - e^{-at_{0.1}} = 0.1 \qquad e^{-at_{0.1}} = 0.9 \qquad t_{0.1} = -\frac{\ln 0.9}{a}$$

Rise Time

Step response: $y(t) = 1(t) - e^{-at}$



Rise time t_r: the time it takes to get from 10% of steady-state value to 90%

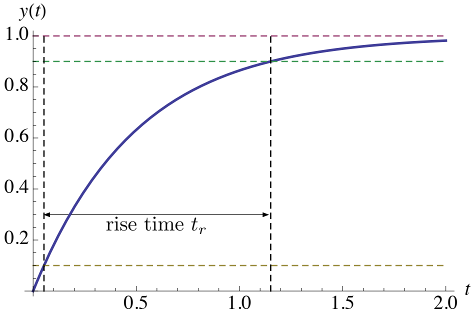
In this example, it is easy to compute t_r analytically:

$$1 - e^{-at_{0.1}} = 0.1 \qquad e^{-at_{0.1}} = 0.9 \qquad t_{0.1} = -\frac{\ln 0.9}{a}$$

$$1 - e^{-at_{0.9}} = 0.9 \qquad e^{-at_{0.9}} = 0.1 \qquad t_{0.9} = -\frac{\ln 0.1}{a}$$

Rise Time

Step response: $y(t) = 1(t) - e^{-at}$



Rise time t_r: the time it takes to get from 10% of steady-state value to 90%

In this example, it is easy to compute t_r analytically:

$$1 - e^{-at_{0.1}} = 0.1 \qquad e^{-at_{0.1}} = 0.9 \qquad t_{0.1} = -\frac{\ln 0.9}{a}$$

$$1 - e^{-at_{0.9}} = 0.9 \qquad e^{-at_{0.9}} = 0.1 \qquad t_{0.9} = -\frac{\ln 0.1}{a}$$

$$t_r = t_{0.9} - t_{0.1} = \frac{\ln 0.9 - \ln 0.1}{a} = \frac{\ln 9}{a} \approx \frac{2.2}{a}$$

Transient Response Specs

Now let's consider the more interesting case: *2nd-order response*

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

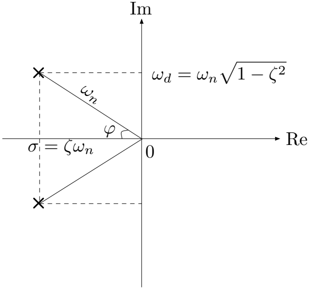
where $\sigma = \zeta\omega_n$ $\omega_d = \omega_n\sqrt{1-\zeta^2}$ $\qquad (\zeta < 1)$

Transient Response Specs

Now let's consider the more interesting case: *2nd-order response*

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

where $\sigma = \zeta\omega_n$ $\omega_d = \omega_n\sqrt{1-\zeta^2}$ $\qquad (\zeta < 1)$

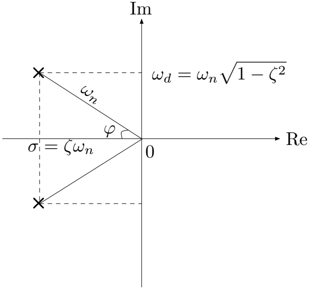


Transient Response Specs

Now let's consider the more interesting case: *2nd-order response*

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

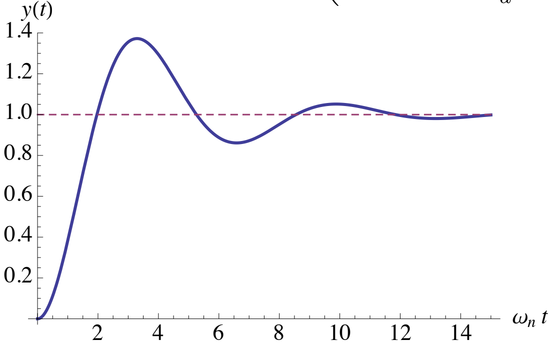
where $\sigma = \zeta\omega_n \; \omega_d = \omega_n\sqrt{1-\zeta^2} \qquad (\zeta < 1)$



Step response: $$y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

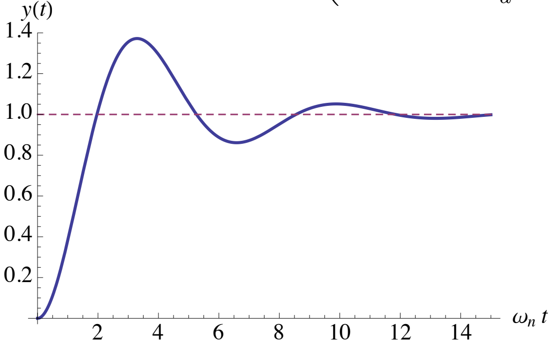
Transient-Response Specs

Step response: $$y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$



Transient-Response Specs

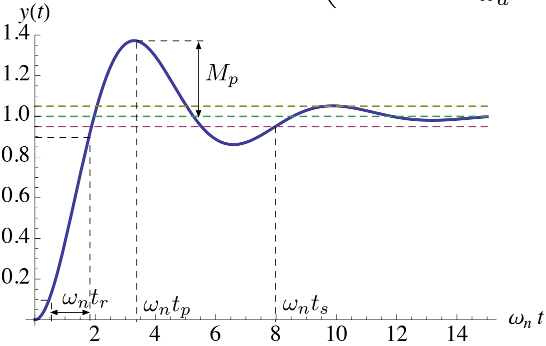
Step response: $$y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$



- rise time t_r — time to get from $0.1y(\infty)$ to $0.9y(\infty)$
- overshoot M_p and peak time t_p
- settling time t_s — first time for transients to decay to within a specified small percentage of $y(\infty)$ and stay in that range (we will usually worry about 5% settling time)

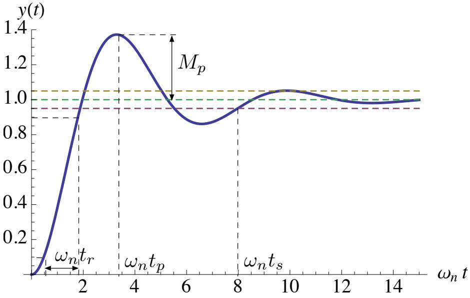
Transient-Response Specs

Step response: $$y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

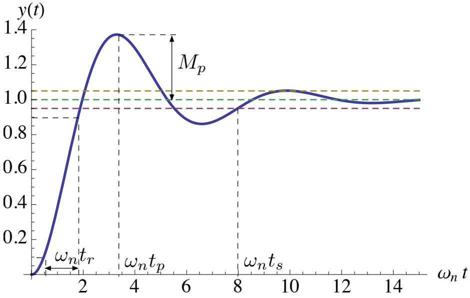


- rise time t_r — time to get from $0.1y(\infty)$ to $0.9y(\infty)$
- overshoot M_p and peak time t_p
- settling time t_s — first time for transients to decay to within a specified small percentage of $y(\infty)$ and stay in that range (we will usually worry about 5% settling time)

Transient-Response (or Time-Domain) Specs

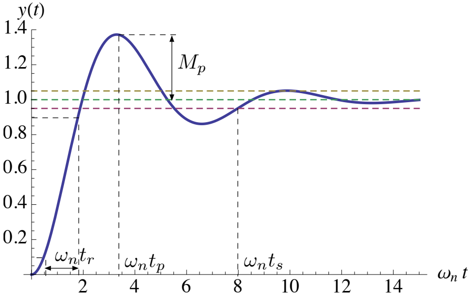


Transient-Response (or Time-Domain) Specs



Do we want these quantities to be large or small?

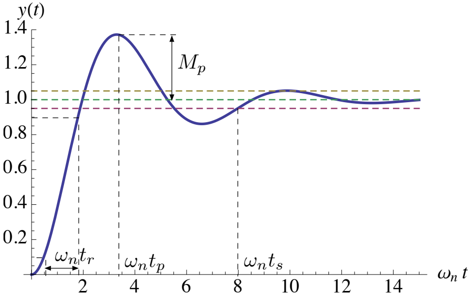
Transient-Response (or Time-Domain) Specs



Do we want these quantities to be large or small?

- t_r

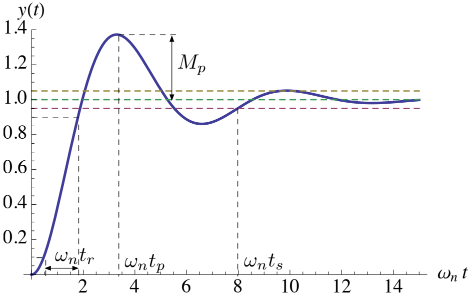
Transient-Response (or Time-Domain) Specs



Do we want these quantities to be large or small?

- t_r small

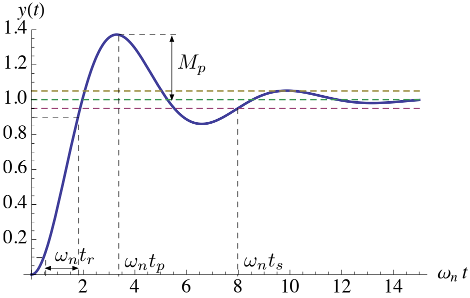
Transient-Response (or Time-Domain) Specs



Do we want these quantities to be large or small?

- t_r small
- M_p

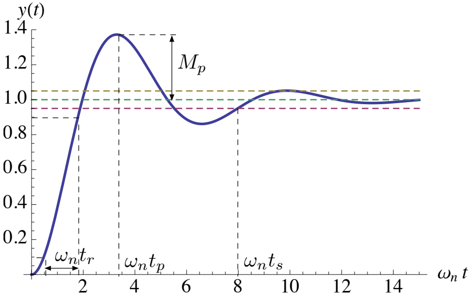
Transient-Response (or Time-Domain) Specs



Do we want these quantities to be large or small?

- t_r small
- M_p small

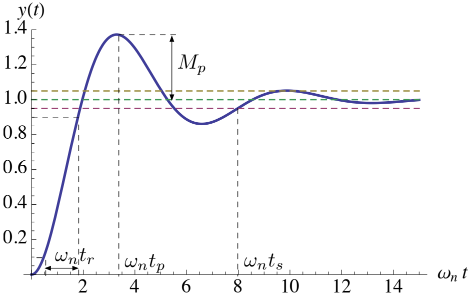
Transient-Response (or Time-Domain) Specs



Do we want these quantities to be large or small?

- t_r small
- M_p small
- t_p

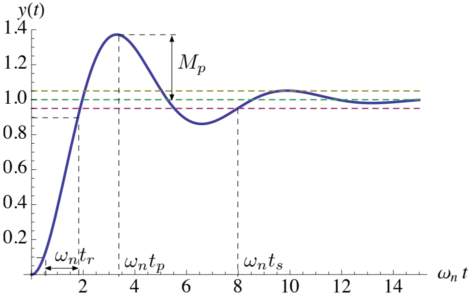
Transient-Response (or Time-Domain) Specs



Do we want these quantities to be large or small?

- t_r small
- M_p small
- t_p small

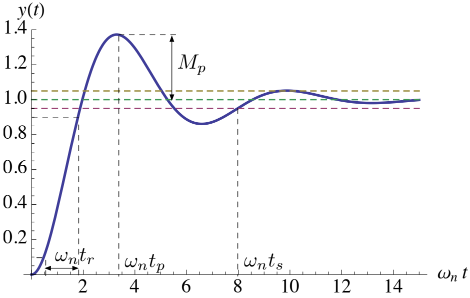
Transient-Response (or Time-Domain) Specs



Do we want these quantities to be large or small?

- t_r small
- M_p small
- t_p small
- t_s

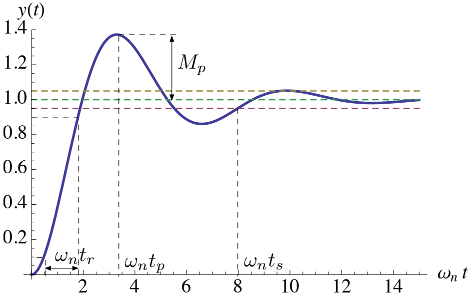
Transient-Response (or Time-Domain) Specs



Do we want these quantities to be large or small?

- t_r small
- M_p small
- t_p small
- t_s small

Transient-Response (or Time-Domain) Specs

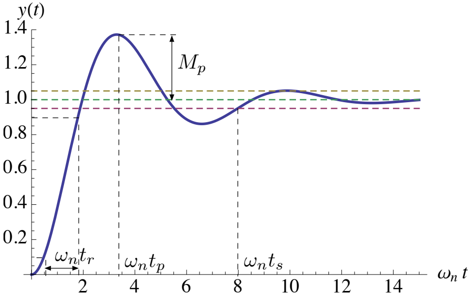


Do we want these quantities to be large or small?

- t_r small
- M_p small
- t_p small
- t_s small

Trade-offs among specs:

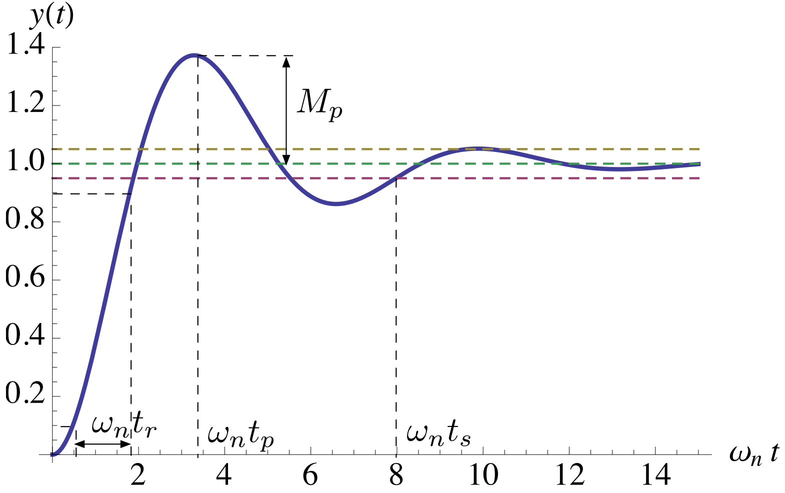
Transient-Response (or Time-Domain) Specs



Do we want these quantities to be large or small?

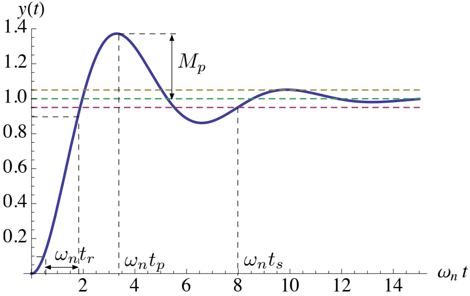
- t_r small
- M_p small
- t_p small
- t_s small

Trade-offs among specs: decrease $t_r \longrightarrow$ increase M_p, etc.

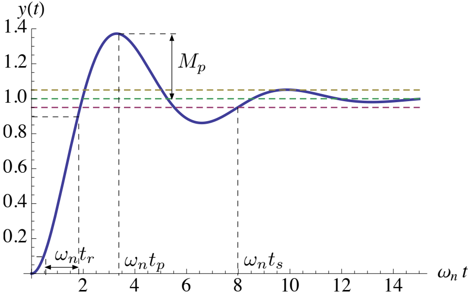


$y(t)$
1.4
1.2
1.0
0.8
0.6
0.4
0.2
M_p
$\omega_n t_r$
$\omega_n t_p$
$\omega_n t_s$
2
4
6
8
10
12
14
$\omega_n\, t$

Formulas for TD Specs: Rise Time

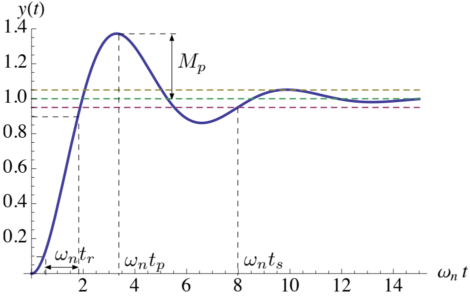


Formulas for TD Specs: Rise Time



Rise time t_r — hard to calculate analytically.

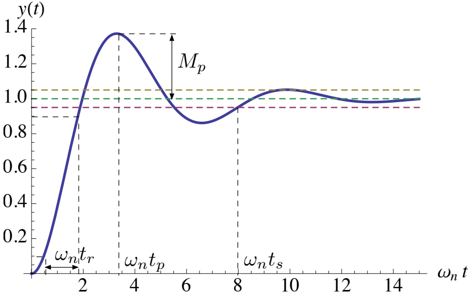
Formulas for TD Specs: Rise Time



Rise time t_r — hard to calculate analytically.

Empirically, on the normalized time scale ($t \to \omega_n t$), rise times are *approximately* the same

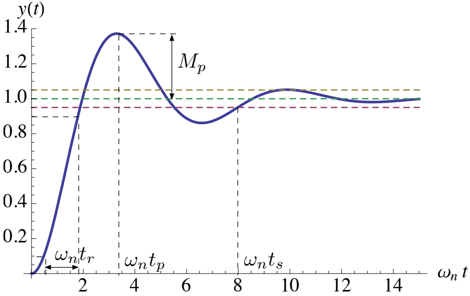
Formulas for TD Specs: Rise Time



Rise time t_r — hard to calculate analytically.
Empirically, on the normalized time scale ($t \to \omega_n t$), rise times are *approximately* the same

$$w_n t_r \approx 1.8 \qquad \text{(exact for } \zeta = 0.5\text{)}$$

Formulas for TD Specs: Rise Time

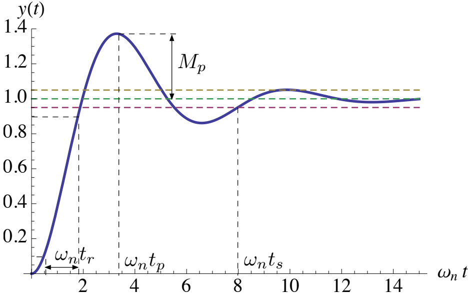


Rise time t_r — hard to calculate analytically.
Empirically, on the normalized time scale ($t \to \omega_n t$), rise times are *approximately* the same

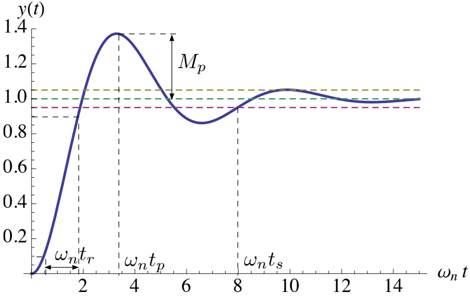
$$w_n t_r \approx 1.8 \qquad \text{(exact for } \zeta = 0.5\text{)}$$

So, we will work with $t_r \approx \dfrac{1.8}{\omega_n}$ (good approx. when $\zeta \approx 0.5$)

Formulas for TD Specs: Overshoot & Peak Time

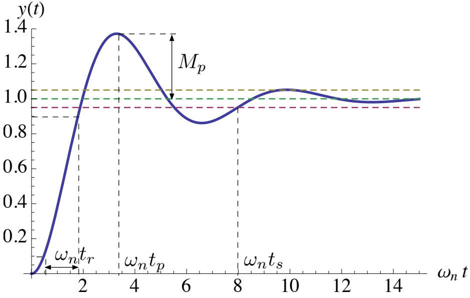


Formulas for TD Specs: Overshoot & Peak Time



t_p is the *first time* $t > 0$ when $y'(t) = 0$

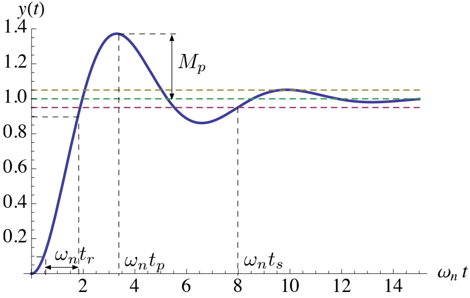
Formulas for TD Specs: Overshoot & Peak Time



t_p is the *first time* $t > 0$ when $y'(t) = 0$

$$y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

Formulas for TD Specs: Overshoot & Peak Time

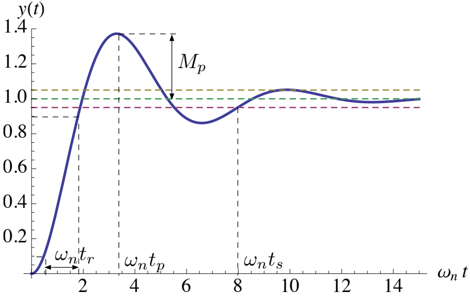


t_p is the *first time* $t > 0$ when $y'(t) = 0$

$$y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

$$y'(t) = \left(\frac{\sigma^2}{\omega_d} + \omega_d\right) e^{-\sigma t}\sin(\omega_d t)$$

Formulas for TD Specs: Overshoot & Peak Time

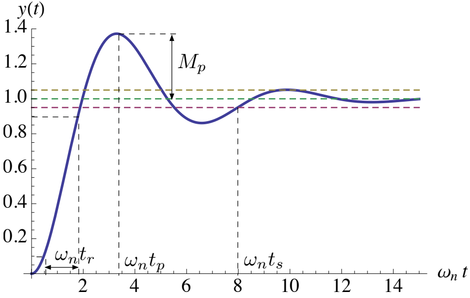


t_p is the *first time* $t > 0$ when $y'(t) = 0$

$$y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

$$y'(t) = \left(\frac{\sigma^2}{\omega_d} + \omega_d\right) e^{-\sigma t}\sin(\omega_d t) = 0 \text{ when } \omega_d t = 0, \pi, 2\pi, \ldots$$

Formulas for TD Specs: Overshoot & Peak Time



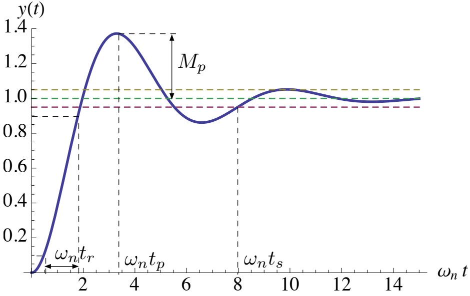
t_p is the *first time* $t > 0$ when $y'(t) = 0$

$$y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

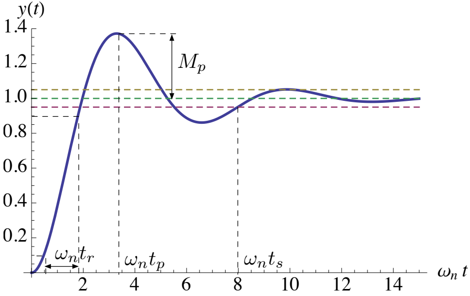
$$y'(t) = \left(\frac{\sigma^2}{\omega_d} + \omega_d\right) e^{-\sigma t}\sin(\omega_d t) = 0 \text{ when } \omega_d t = 0, \pi, 2\pi, \ldots$$

so $t_p = \dfrac{\pi}{\omega_d}$

Formulas for TD Specs: Overshoot & Peak Time

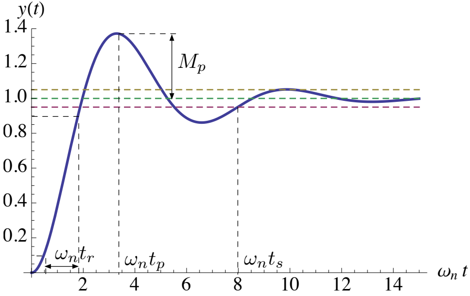


Formulas for TD Specs: Overshoot & Peak Time



We have just computed $t_p = \dfrac{\pi}{\omega_d}$

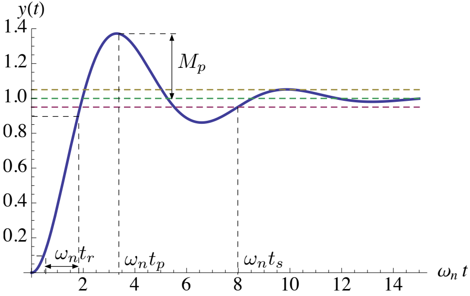
Formulas for TD Specs: Overshoot & Peak Time



We have just computed $t_p = \dfrac{\pi}{\omega_d}$

To find M_p, plug this value into $y(t)$:

Formulas for TD Specs: Overshoot & Peak Time

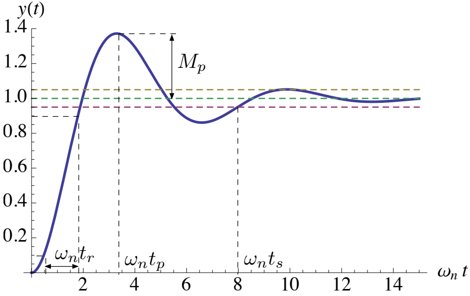


We have just computed $t_p = \dfrac{\pi}{\omega_d}$

To find M_p, plug this value into $y(t)$:

$$M_p = y(t_p) - 1$$

Formulas for TD Specs: Overshoot & Peak Time

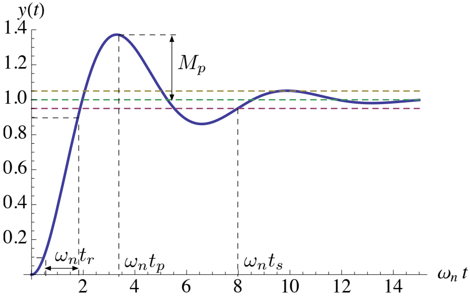


We have just computed $t_p = \dfrac{\pi}{\omega_d}$

To find M_p, plug this value into $y(t)$:

$$M_p = y(t_p) - 1 = -e^{-\frac{\sigma\pi}{\omega_d}} \left(\cos\left(\omega_d \frac{\pi}{\omega_d}\right) + \frac{\sigma}{\omega_d} \sin\left(\omega_d \frac{\pi}{\omega_d}\right)\right)$$

Formulas for TD Specs: Overshoot & Peak Time



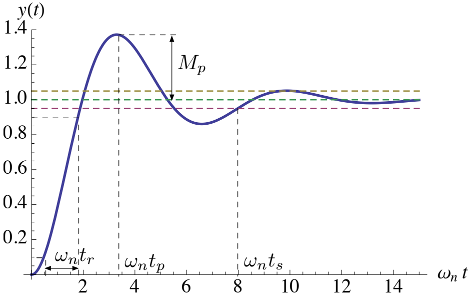
We have just computed $t_p = \dfrac{\pi}{\omega_d}$

To find M_p, plug this value into $y(t)$:

$$M_p = y(t_p) - 1 = -e^{-\frac{\sigma\pi}{\omega_d}} \left(\cos\left(\omega_d \frac{\pi}{\omega_d}\right) + \frac{\sigma}{\omega_d} \sin\left(\omega_d \frac{\pi}{\omega_d}\right)\right)$$

$$= \exp\left(-\frac{\sigma\pi}{\omega_d}\right)$$

Formulas for TD Specs: Overshoot & Peak Time



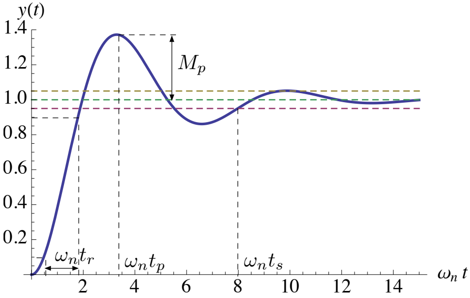
We have just computed $t_p = \frac{\pi}{\omega_d}$

To find M_p, plug this value into $y(t)$:

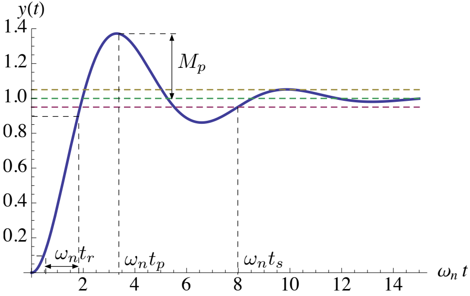
$$M_p = y(t_p) - 1 = -e^{-\frac{\sigma\pi}{\omega_d}} \left(\cos\left(\omega_d \frac{\pi}{\omega_d}\right) + \frac{\sigma}{\omega_d} \sin\left(\omega_d \frac{\pi}{\omega_d}\right)\right)$$

$$= \exp\left(-\frac{\sigma\pi}{\omega_d}\right) = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) \quad \text{— exact formula}$$

Formulas for TD Specs: Settling Time

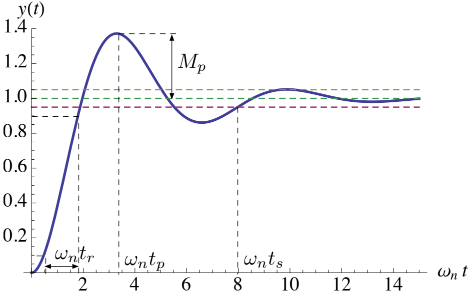


Formulas for TD Specs: Settling Time



$$t_s = \min\left\{t > 0 : \frac{|y(t') - y(\infty)|}{y(\infty)} \le 0.05 \text{ for all } t' \ge t\right\}$$ (here, $y(\infty) = 1$)

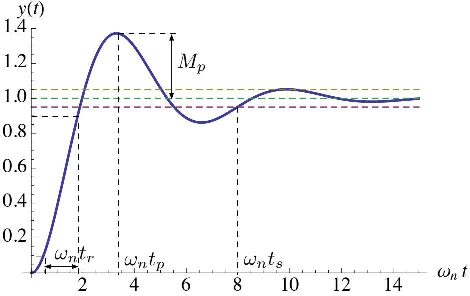
Formulas for TD Specs: Settling Time



$$t_s = \min \left\{ t > 0 : \frac{|y(t') - y(\infty)|}{y(\infty)} \leq 0.05 \text{ for all } t' \geq t \right\}$$ (here, $y(\infty) = 1$)

$$|y(t) - 1| = e^{-\sigma t} \left| \cos(\omega_d t) + \frac{\sigma}{\omega_d} \sin(\omega_d t) \right|$$

Formulas for TD Specs: Settling Time

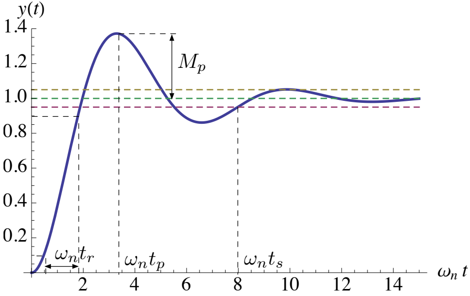


$t_s = \min\left\{ t > 0 : \dfrac{|y(t') - y(\infty)|}{y(\infty)} \leq 0.05 \text{ for all } t' \geq t \right\}$ (here, $y(\infty) = 1$)

$$|y(t) - 1| = e^{-\sigma t} \left| \cos(\omega_d t) + \frac{\sigma}{\omega_d} \sin(\omega_d t) \right|$$

here, $e^{-\sigma t}$ is what matters (sin and cos are bounded between ± 1), so $e^{-\sigma t_s} \leq 0.05$

Formulas for TD Specs: Settling Time



$t_s = \min \left\{ t > 0 : \frac{|y(t') - y(\infty)|}{y(\infty)} \leq 0.05 \text{ for all } t' \geq t \right\}$ (here, $y(\infty) = 1$)

$$|y(t) - 1| = e^{-\sigma t} \left| \cos(\omega_d t) + \frac{\sigma}{\omega_d} \sin(\omega_d t) \right|$$

here, $e^{-\sigma t}$ is what matters (sin and cos are bounded between ± 1), so $e^{-\sigma t_s} \leq 0.05$ this gives $t_s = -\frac{\ln 0.05}{\sigma} \approx \frac{3}{\sigma}$

Formulas for TD Specs

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\sigma^2 + \omega_d^2}{(s+\sigma)^2 + \omega_d^2}$$

$$t_r \approx \frac{1.8}{\omega_n}$$

$$t_p = \frac{\pi}{\omega_d}$$

$$M_p = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

$$t_s \approx \frac{3}{\sigma}$$

TD Specs in Frequency Domain

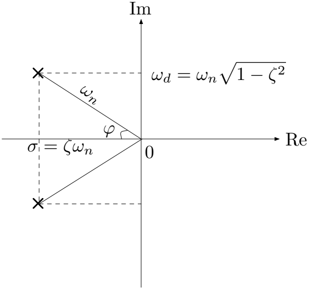
We want to *visualize* time-domain specs in terms of *admissible pole locations* for the 2nd-order system

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\sigma^2 + \omega_d^2}{(s+\sigma)^2 + \omega_d^2}$$

where $\sigma = \zeta\omega_n$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Step response: $y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$



$$\omega_n^2 = \sigma^2 + \omega_d^2$$
$$\zeta = \cos\varphi$$

Rise Time in Frequency Domain

Suppose we want $t_r \leq c$ (c is some desired given value)

Rise Time in Frequency Domain

Suppose we want $t_r \le c$ (c is some desired given value)

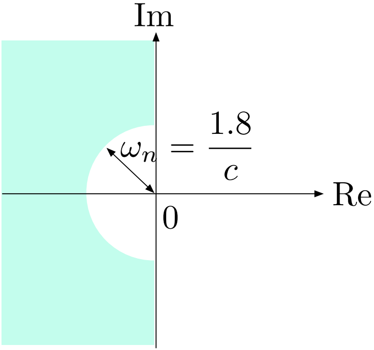
$$t_r \approx \frac{1.8}{\omega_n} \le c \qquad \Longrightarrow \qquad \omega_n \ge \frac{1.8}{c}$$

Rise Time in Frequency Domain

Suppose we want $t_r \leq c$ (c is some desired given value)

$$t_r \approx \frac{1.8}{\omega_n} \leq c \qquad \Longrightarrow \qquad \omega_n \geq \frac{1.8}{c}$$

Geometrically, we want poles to lie in the shaded region:



(recall that ω_n is the *magnitude of the poles*)

Overshoot in Frequency Domain

Suppose we want $M_p \leq c$

Overshoot in Frequency Domain

Suppose we want $M_p \leq c$

$$M_p = \underbrace{\exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}_{\text{decreasing function}} \leq c$$

Overshoot in Frequency Domain

Suppose we want $M_p \leq c$

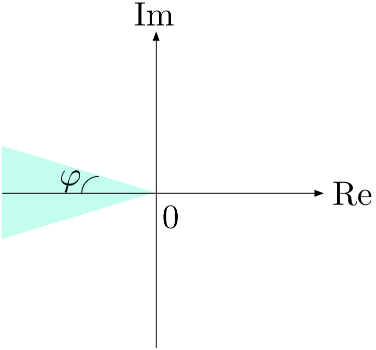
$$M_p = \underbrace{\exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}_{\text{decreasing function}} \leq c \qquad \text{— need large damping ratio}$$

Overshoot in Frequency Domain

Suppose we want $M_p \leq c$

$$M_p = \underbrace{\exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}_{\text{decreasing function}} \leq c \qquad \text{— need large damping ratio}$$

Geometrically, we want poles to lie in the shaded region:

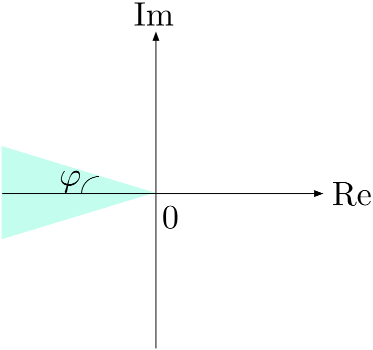


Overshoot in Frequency Domain

Suppose we want $M_p \leq c$

$$M_p = \underbrace{\exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}_{\text{decreasing function}} \leq c \qquad \text{— need large damping ratio}$$

Geometrically, we want poles to lie in the shaded region:



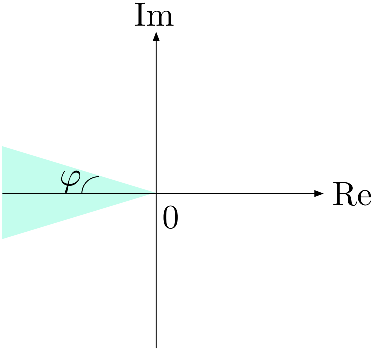
$$\frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{\omega_n \zeta}{\omega_n \sqrt{1-\zeta^2}}$$
$$= \frac{\sigma}{\omega_d} = \cot\varphi$$

Overshoot in Frequency Domain

Suppose we want $M_p \leq c$

$$M_p = \underbrace{\exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}_{\text{decreasing function}} \leq c \qquad \text{— need large damping ratio}$$

Geometrically, we want poles to lie in the shaded region:



$$\frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{\omega_n \zeta}{\omega_n \sqrt{1-\zeta^2}}$$
$$= \frac{\sigma}{\omega_d} = \cot\varphi$$

— need φ to be small

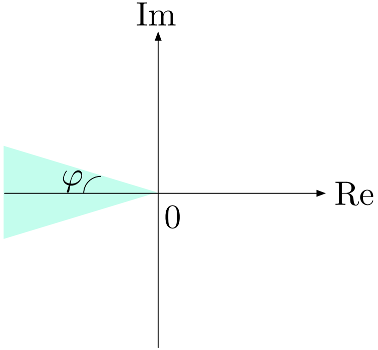
Overshoot in Frequency Domain

Suppose we want $M_p \leq c$

$$M_p = \underbrace{\exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}_{\text{decreasing function}} \leq c$$

— need large damping ratio

Geometrically, we want poles to lie in the shaded region:



$$\frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{\omega_n\zeta}{\omega_n\sqrt{1-\zeta^2}}$$
$$= \frac{\sigma}{\omega_d} = \cot\varphi$$

— need φ to be small

Intuition: good damping $\rightarrow$ good decay in 1/2 period

Settling Time in Frequency Domain

Suppose we want $t_s \leq c$

Settling Time in Frequency Domain

Suppose we want $t_s \leq c$

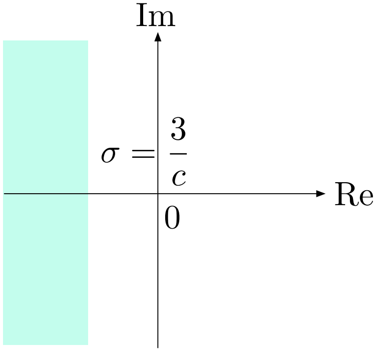
$$t_s \approx \frac{3}{\sigma} \leq c \qquad \Longrightarrow \qquad \sigma \geq \frac{3}{c}$$

Settling Time in Frequency Domain

Suppose we want $t_s \leq c$

$$t_s \approx \frac{3}{\sigma} \leq c \qquad \Longrightarrow \qquad \sigma \geq \frac{3}{c}$$

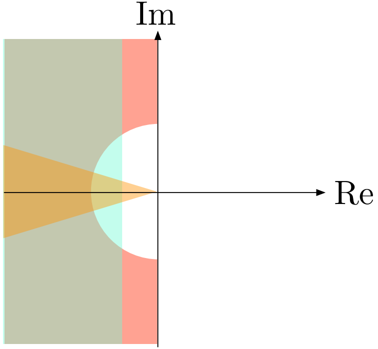
Want poles to be sufficiently fast (large enough magnitude of real part):



Intuition: poles far to the left → transients decay faster → smaller t_s

Combination of Specs

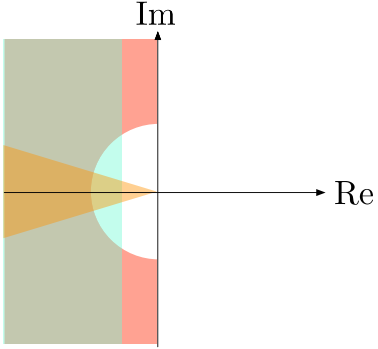
If we have specs for any combination of t_r, M_p, t_s, we can easily relate them to allowed pole locations:



The shape and size of the region for admissible pole locations will change depending on which specs are more severely constrained.

Combination of Specs

If we have specs for any combination of t_r, M_p, t_s, we can easily relate them to allowed pole locations:

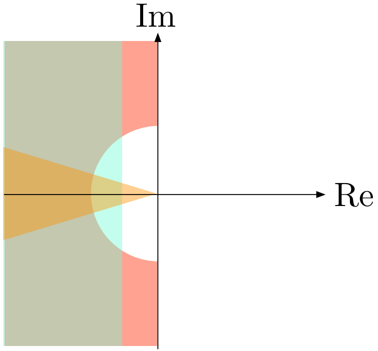


The shape and size of the region for admissible pole locations will change depending on which specs are more severely constrained.

This is very appealing to engineers: easy to visualize things, no such crisp visualization in time domain.

Combination of Specs

If we have specs for any combination of t_r, M_p, t_s, we can easily relate them to allowed pole locations:



The shape and size of the region for admissible pole locations will change depending on which specs are more severely constrained.

This is very appealing to engineers: easy to visualize things, no such crisp visualization in time domain.

But: not very rigorous, and also only valid for our prototype 2nd-order system, which has only 2 poles and no zeros ...