# Plan of the Lecture

- Review: control design using frequency response
- Today's topic: Nyquist stability criterion

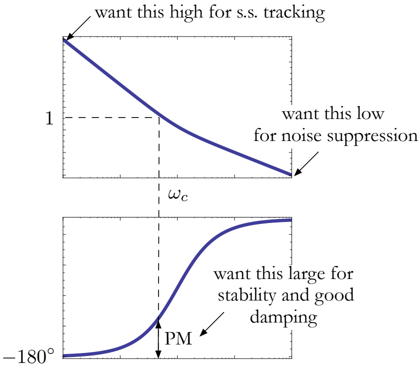
*Goal:* learn how to detect the presence of RHP poles of the closed-loop transfer function as the gain $K$ is varied using frequency-response data

*Reading:* FPE, Chapter 6

# Review: Frequency Domain Design Method

Design based on Bode plots is good for:

- easily visualizing the concepts




- evaluating the design and seeing which way to change it
- using experimental data (frequency response of the uncontrolled system can be measured experimentally)

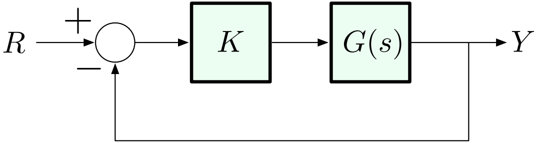
# Review: Frequency Domain Design Method

Design based on Bode plots is not good for:

- exact closed-loop pole placement (root locus is more suitable for that)
- deciding if a given $K$ is stabilizing or not ...
  - we can only measure *how far* we are from instability (using GM or PM), if we know that we are stable
  - however, we don't have a way of checking whether a given $K$ is stabilizing from frequency response data

What we want is a frequency-domain substitute for the Routh–Hurwitz criterion — this is the Nyquist criterion, which we will discuss in today's lecture.

# Nyquist Stability Criterion




Goal: count the number of RHP poles (if any) of the closed-loop transfer function

$$\frac{KG(s)}{1 + KG(s)}$$

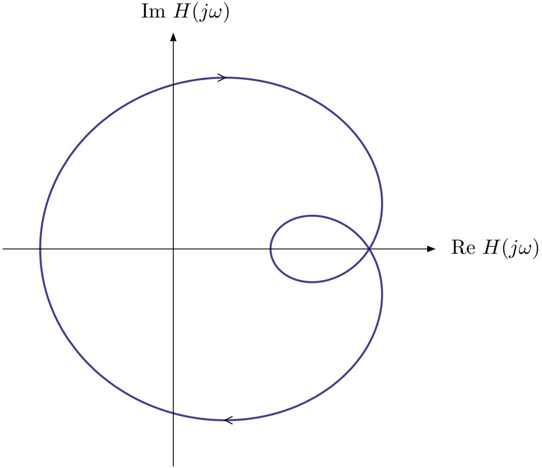
based on frequency-domain characteristics of the plant transfer function $G(s)$

# Review: Nyquist Plot

Consider an arbitrary *strictly proper* transfer function $H$:

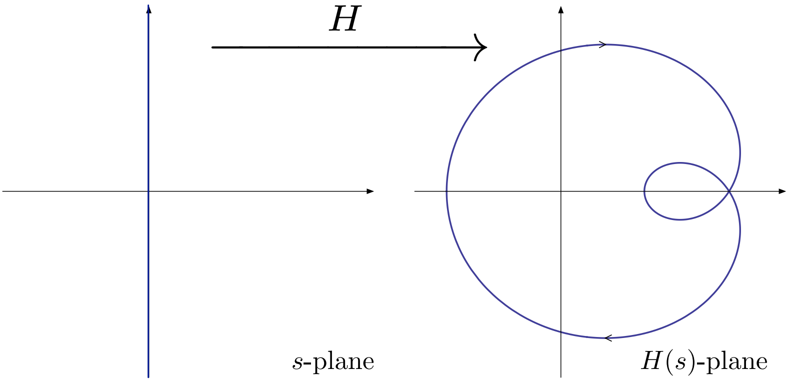
$$H(s) = \frac{(s - z_1) \dots (s - z_m)}{(s - p_1) \dots (s - p_n)}, \qquad m < n$$

Nyquist plot: $\operatorname{Im} H(j\omega)$ vs. $\operatorname{Re} H(j\omega)$ as $\omega$ varies from $-\infty$ to $\infty$



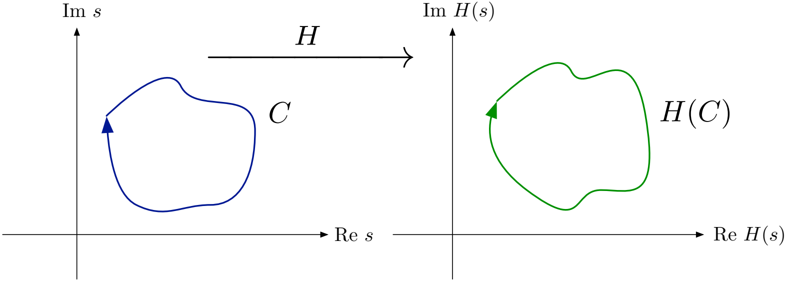
# Nyquist Plot as a Mapping of the $s$-Plane

We can view the Nyquist plot of $H$ as the image of the imaginary axis $\{j\omega : -\infty < \omega < \infty\}$ under the mapping $H : \mathbb{C} \to \mathbb{C}$



# Transformation of a Closed Contour Under $H$

If we choose any closed curve (or *contour*) $C$ on the left, it will get mapped by $H$ to some other curve (contour) on the right:




Important: when working with contours in the complex plane, always keep track of the direction in which we traverse the contour (clockwise vs. counterclockwise)!!

# Phase of $H$ Along a Contour

For any $s \in \mathbb{C}$, the phase (or *argument*) of $H(s)$ is

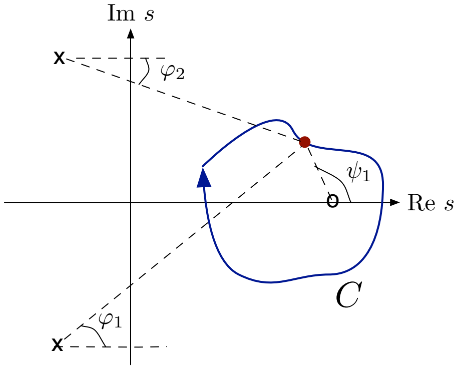
$$\begin{aligned} \angle H(s) &= \angle \frac{(s-z_1)\ldots(s-z_m)}{(s-p_1)\ldots(s-p_n)} \\ &= \sum_{i=1}^{m} \angle(s-z_i) - \sum_{j=1}^{n} \angle(s-p_j) \\ &= \sum_{i=1}^{m} \psi_i - \sum_{j=1}^{n} \varphi_j \end{aligned}$$

We are interested in how $\angle H(s)$ changes as $s$ traverses a closed, clockwise ($\circlearrowright$) oriented contour $C$ in the complex plane.

We will look at several cases, depending on how the contour is located relative to poles and zeros of $H$.

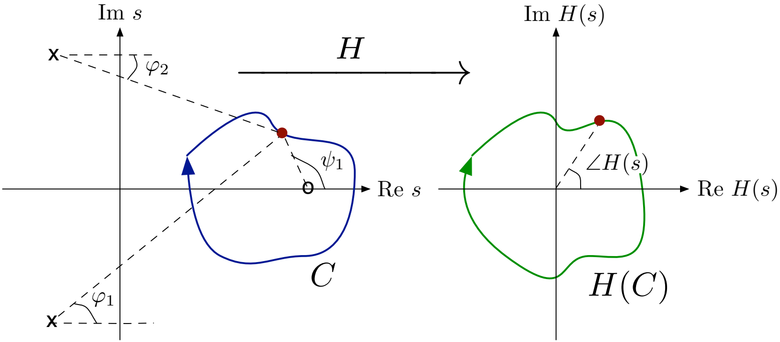
# Case 1: Contour Encircles a Zero

Suppose that $C$ is a closed, ↻-oriented contour in $\mathbb{C}$ that encircles a zero of $H(s)$:




How does $\angle H(s)$ change as we go around $C$?

# Case 1: Contour Encircles a Zero



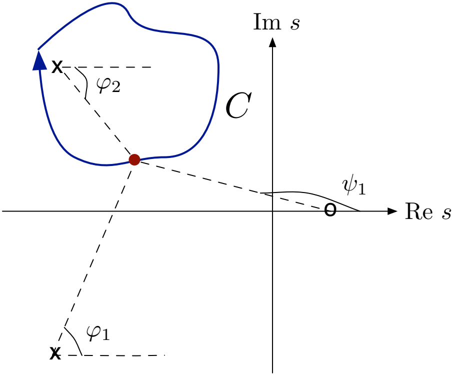

How does $\angle H(s)$ change as we go around $C$?

Let's see what happens to angles from $s$ to poles/zeros of $H$:

- $\varphi_1$ and $\varphi_2$ return to their original values
- $\psi_1$ picks up a net change of $-360°$
- therefore, $\angle H(s)$ picks up a net change of $-360°$, so $H(C)$ encircles the origin once, clockwise (↻)

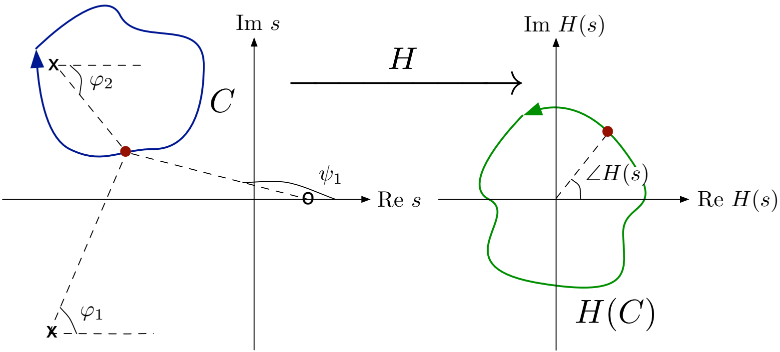
# Case 2: Contour Encircles a Pole

Suppose that $C$ is a closed, ↻-oriented contour in $\mathbb{C}$ that encircles a pole of $H(s)$:




How does $\angle H(s)$ change as we go around $C$?

# Case 2: Contour Encircles a Pole



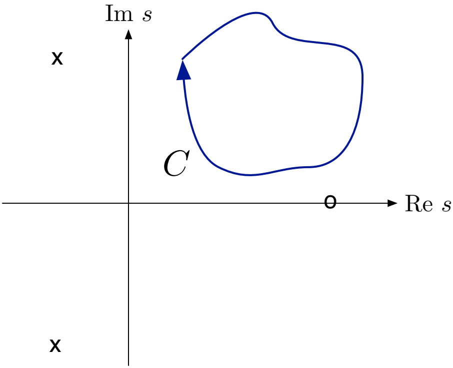

How does $\angle H(s)$ change as we go around $C$?

Let's see what happens to angles from $s$ to poles/zeros of $H$:

- $\varphi_1$ and $\psi_1$ return to their original values
- $\varphi_2$ picks up a net change of $-360°$
- therefore, $\angle H(s)$ picks up a net change of $360°$, so $H(C)$ encircles the origin once counterclockwise (↺)

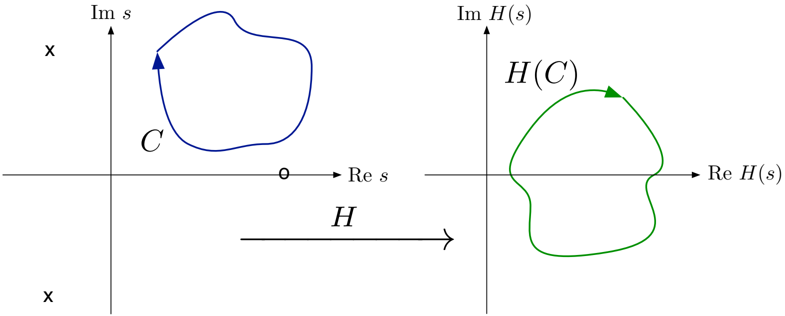
# Case 3: Contour Encircles No Poles or Zeros

Suppose that $C$ is a closed, $\circlearrowright$-oriented contour in $\mathbb{C}$ that does not encircle any poles or zeros of $H(s)$:




How does $\angle H(s)$ change as we go around $C$?

# Case 3: Contour Encircles No Poles or Zeros




How does $\angle H(s)$ change as we go around $C$?

Let's see what happens to angles from $s$ to poles/zeros of $H$:

- $\varphi_1, \varphi_2, \psi_1$ all return to their original values
- therefore, no net change in $\angle H(s)$, so $H(C)$ does not encircle the origin

# The Argument Principle

These special cases all lead to the following general result:

The Argument Principle. Let $C$ be a closed, clockwise $\circlearrowright$ oriented contour not passing through any zeros or poles* of $H(s)$. Let $H(C)$ be the image of $C$ under the map $s \mapsto H(s)$:

$$H(C) = \{H(s) : s \in \mathbb{C}\}.$$

Then:

$$\begin{aligned}&\#(\text{clockwise encirclements } \circlearrowright \text{ of } 0 \text{ by } H(C))\\ &= \#(\text{zeros of } H(s) \text{ inside } C) - \#(\text{poles of } H(S) \text{ inside } C).\end{aligned}$$

More succinctly,

$$N = Z - P$$

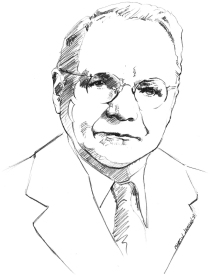
* will see the reason for this later ...

# The Argument Principle

$$N = Z - P$$

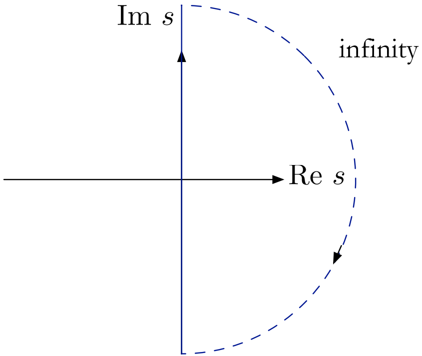
- If $N < 0$, it means that $H(C)$ encircles the origin counterclockwise ($\circlearrowleft$).
- We do not want $C$ to pass through any pole of $H$ because then $H(C)$ would not be defined.
- We also do not want $C$ to pass through any zero of $H$ because then $0 \in H(C)$, so #(encirclements) is not well-defined.

# From Argument Principle to Nyquist Criterion



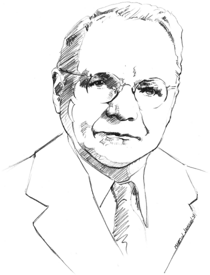
Harry Nyquist
(1889–1976)

- We are interested in RHP poles, so let's choose a suitable contour $C$ that *encloses the RHP*:




- From now on, $C$ = imaginary axis plus the "path around infinity."
- If $H$ is strictly proper, then $H(\infty) = 0$.

# From Argument Principle to Nyquist Criterion



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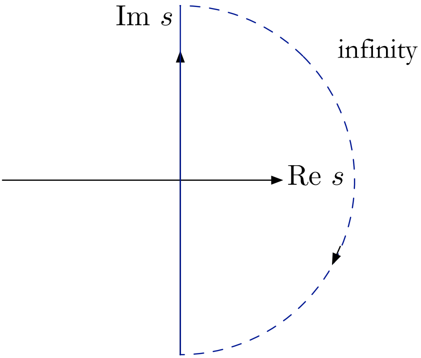
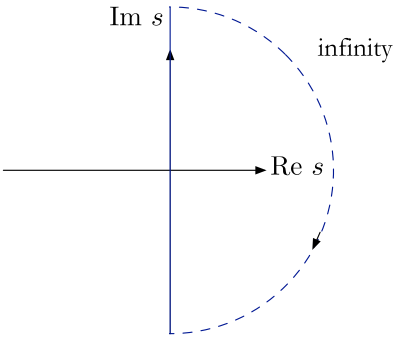



With this choice of $C$,

$$H(C) = \text{Nyquist plot of } H$$

(image of the imaginary axis under the map $H : \mathbb{C} \to \mathbb{C}$; if $H$ is strictly proper, $0 = H(\infty)$)

# From Argument Principle to Nyquist Criterion

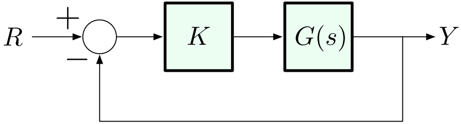



$$H(C) = \text{Nyquist plot of } H$$

We are interested in RHP roots of $1 + KG(s)$, where $G$ is the plant transfer function.

Thus, we choose $H(s) = 1 + KG(s)$

# From Argument Principle to Nyquist Criterion




We now examine the Nyquist plot of $H(s) = 1 + KG(s)$.

By the argument principle,

$$N = Z - P,$$

$$\text{where } N = \#(\circlearrowright \text{ encirclements of } 0 \text{ by Nyquist plot of } 1 + KG(s)),$$

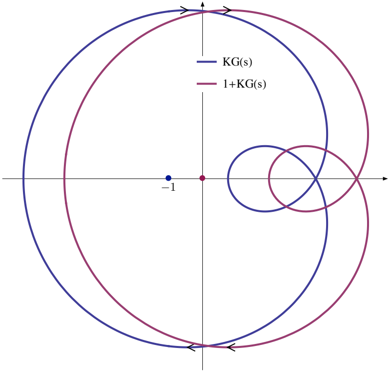
$$Z = \#(\text{zeros of } 1 + KG(s) \text{ inside } C),$$

$$P = \#(\text{poles of } 1 + KG(s) \text{ inside } C)$$

Now we extract information about RHP roots of $1 + KG(s)$

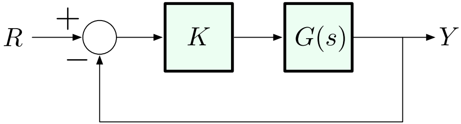
# Nyquist Criterion: $N$

$$\begin{aligned} N &= \#(\circlearrowright \text{ encirclements of } 0 \text{ by Nyquist plot of } 1 + KG(s)) \\ &= \#(\circlearrowright \text{ encirclements of } -1 \text{ by Nyquist plot of } KG(s)) \\ &= \#(\circlearrowright \text{ encirclements of } -1/K \text{ by Nyquist plot of } G(s)) \end{aligned}$$




— can be read off the Nyquist plot of the *open-loop* t.f. $G$!!

# Nyquist Criterion: $Z$




$$G(s) = \frac{q(s)}{p(s)}, \qquad \deg(q) \le \deg(p)$$

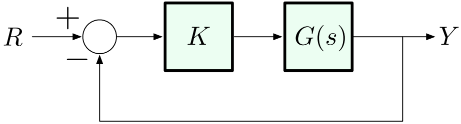
$$1 + KG(s) = \frac{p(s) + Kq(s)}{p(s)}$$

$$\text{closed-loop t.f.} = \frac{KG(s)}{1 + KG(s)} = \frac{Kq(s)}{p(s) + Kq(s)}$$

Therefore:

$$\begin{aligned} Z &= \#(\text{zeros of } 1 + KG(s) \text{ inside } C) \\ &= \#(\text{RHP zeros of } 1 + KG(s)) \\ &= \#(\text{RHP closed-loop poles}) \end{aligned}$$

# Nyquist Criterion: $P$



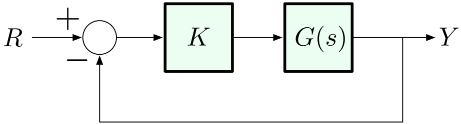

$$G(s) = \frac{q(s)}{p(s)}, \qquad \deg(q) \leq \deg(p)$$

$$1 + KG(s) = 1 + K\frac{q(s)}{p(s)} = \frac{p(s) + Kq(s)}{p(s)}$$

Therefore:

$$\begin{aligned} P &= \#(\text{poles of } 1 + KG(s) \text{ inside } C) \\ &= \#(\text{RHP poles of } 1 + KG(s)) \\ &= \#(\text{RHP roots of } p(s)) \\ &= \#(\text{RHP open-loop poles}) \end{aligned}$$

# The Nyquist Theorem



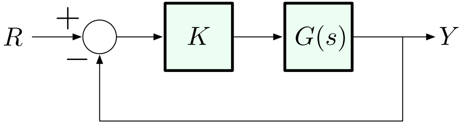

Nyquist Theorem (1928) Assume that $G(s)$ has no poles on the imaginary axis*, and that its Nyquist plot does not pass through the point $-1/K$. Then

$$N = Z - P$$

$$\#(\circlearrowright \text{ of } -1/K \text{ by Nyquist plot of } G(s))$$
$$= \#(\text{RHP closed-loop poles}) - \#(\text{RHP open-loop poles})$$

* Easy to fix: draw an infinitesimally small circular path that goes *around* the pole and stays in RHP

# The Nyquist Stability Criterion




$$\underbrace{N}_{\#(\circlearrowright \text{ of } -1/K)} = \underbrace{Z}_{\#(\text{unstable CL poles})} - \underbrace{P}_{\#(\text{unstable OL poles})}$$

$$Z = N + P$$

$$Z = 0 \quad \Longrightarrow N = -P$$

Nyquist Stability Criterion. Under the assumptions of the Nyquist theorem, the closed-loop system (at a given gain $K$) is stable *if and only if* the Nyquist plot of $G(s)$ encircles the point $-1/K$ $P$ times *counterclockwise*, where $P$ is the number of unstable (RHP) open-loop poles of $G(s)$.

# Applying the Nyquist Criterion

Workflow:

$$\text{Bode } M \text{ and } \phi\text{-plots} \quad \longrightarrow \quad \text{Nyquist plot}$$

### Advantages of Nyquist over Routh–Hurwitz

- can work directly with experimental frequency response data (e.g., if we have the Bode plot based on measurements, but do not know the transfer function)
- less computational, more geometric (came 55 years after Routh)

# Example

$$G(s) = \frac{1}{(s+1)(s+2)} \qquad \text{(no open-loop RHP poles)}$$

Characteristic equation:

$$(s+1)(s+2) + K = 0 \qquad \Longleftrightarrow \qquad s^2 + 3s + K + 2 = 0$$

From Routh, we already know that the closed-loop system is stable for $K > -2$.

We will now reproduce this answer using the Nyquist criterion.

# Example

$$G(s) = \frac{1}{(s+1)(s+2)} \qquad \text{(no open-loop RHP poles)}$$

Strategy:

- Start with the Bode plot of $G$
- Use the Bode plot to graph $\mathrm{Im}\, G(j\omega)$ vs. $\mathrm{Re}\, G(j\omega)$ for $0 \le \omega < \infty$
- This gives only a *portion* of the entire Nyquist plot

$$\left(\mathrm{Re}\, G(j\omega), \mathrm{Im}\, G(j\omega)\right), \qquad -\infty < \omega < \infty$$

- Symmetry:

$$G(-j\omega) = \overline{G(j\omega)}$$

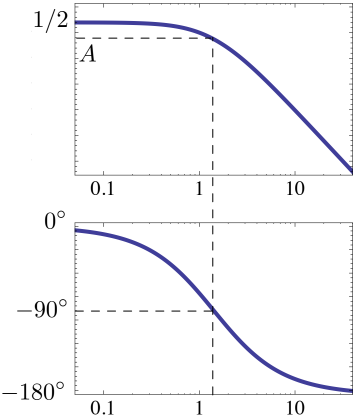
— Nyquist plots are always *symmetric w.r.t. the real axis*!!

# Example

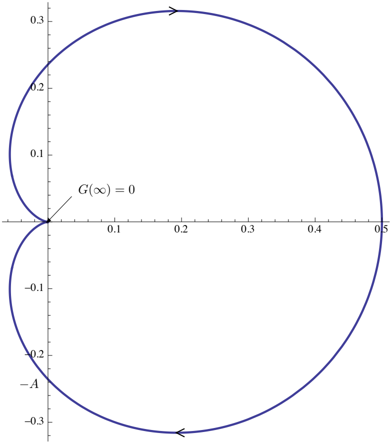
$$G(s) = \frac{1}{(s+1)(s+2)}$$

(no open-loop RHP poles)

Bode plot:



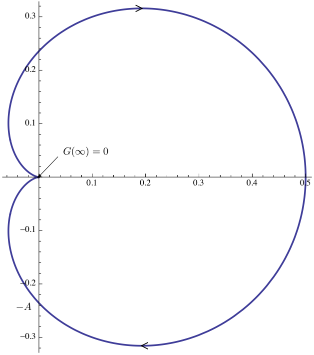

Nyquist plot:



# Example: Applying the Nyquist Criterion

$$G(s) = \frac{1}{(s+1)(s+2)} \qquad \text{(no open-loop RHP poles)}$$

Nyquist plot:




$$\#(\circlearrowright \text{ of } -1/K) = \#(\text{RHP CL poles}) - \underbrace{\#(\text{RHP OL poles})}_{=0}$$

$\Longrightarrow K \in \mathbb{R}$ is stabilizing if and only if

$$\#(\circlearrowright \text{ of } -1/K) = 0$$

- If $K > 0$, $\#(\circlearrowright \text{ of } -1/K) = 0$
- If $0 < -1/K < 1/2$, $\#(\circlearrowright \text{ of } -1/K) > 0 \Longrightarrow$ closed-loop stable for $K > -2$