Plan of the Lecture

- **Review:** basic properties and benefits of feedback control
- **Today's topic:** introduction to Proportional-Integral-Derivative (PID) control

Goal: study basic features and capabilities of PID control (industry standard since 1950's): arbitrary pole placement; reference tracking; disturbance rejection

Reading: FPE, Sections 4.1–4.3; lab manual

Recap: Benefits of Feedback Control

From last lecture: feedback control

- reduces steady-state error to disturbances
- reduces steady-state sensitivity to model uncertainty (parameter variations)
- improves time response

So far, we have only looked at *proportional feedback* (scalar gain) and 1st-order plants. Now we will add two more basic ingredients and examine their effect on higher-order systems.

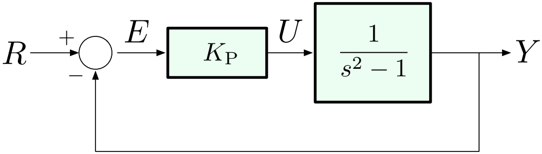
We will consider the following plant transfer function:

$$G(s) = \frac{1}{s^2 - 1}$$

- unstable: poles at $s = \pm 1$ (one pole in RHP)
- 2nd-order

– not as easy as DC motor, which was 1st-order and stable.

Proportional Feedback



K_P – "proportional gain" (P-gain) $\qquad U = K_P E$

Let's try to find a value of K_P that would stabilize the system:

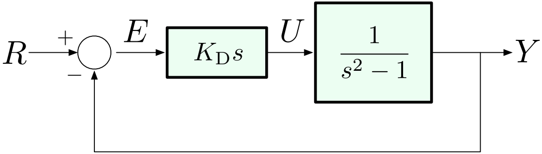
$$\frac{Y}{R} = \frac{\frac{K_P}{s^2-1}}{1+\frac{K_P}{s^2-1}} = \frac{K_P}{s^2-1+K_P}$$

— the polynomial in the denominator has zero coefficient of s
$\Longrightarrow$ necessary condition for stability is not satisfied.

The feedback system is *not stable for **any** value of* K_P*!!*

Derivative Feedback

Let's feed the *derivative of the error*, multiplied by some gain, back into the plant:



Motivation: derivative = rate of change; faster change $\Longrightarrow$ more control needed.

Caveat: multiplication by s is not a causal element (why?)

Derivative action and lack of causality: recall

$$\dot{e}(t) \approx \frac{e(t+\delta) - e(t)}{\delta} \qquad \text{(for small } \delta\text{)}$$

— if $\delta > 0$, $e(t+\delta)$ is in the future of $e(t)$!!

Disclaimer 1 about D-Feedback: Lack of Causality

Consider some state-space models:

$$\dot{x} = Ax + Bu \qquad sX = AX + BU \qquad (s-A)X = BU$$
$$y = Cx \qquad Y = CX \qquad \frac{Y}{U} = \frac{CB}{s-A} \equiv \frac{q(s)}{p(s)}$$

$\deg(q) < \deg(p)$ — strictly proper transfer function

$$\dot{x} = Ax + Bu \qquad sX = AX + BU \qquad (s-A)X = BU$$
$$y = Cx + Du \qquad Y = CX + DU \qquad Y = \frac{CB}{s-A}U + DU$$
$$= \frac{CB + D(s-A)}{s-A}U \equiv \frac{q(s)}{p(s)}$$

$\deg(q) = \deg(p)$ — proper transfer function

Causal systems have proper transfer functions.

Lack of Causality

But if $u = K\dot{e}$, then $U = KsE \implies \frac{U}{E} = Ks = \frac{q(s)}{p(s)}$

$\deg(q) > \deg(p)$— *improper system* (lack of causality)

So, $E \mapsto K_{\mathrm{D}}sE$ is not implementable directly, but we can implement an approximation, e.g.

$$\frac{K_{\mathrm{D}}as}{a+s} \longrightarrow K_{\mathrm{D}}s \qquad \text{as } a \to \infty$$

(this can be done using op-amps).

Alternatively, we can approximate derivative action using finite differences:

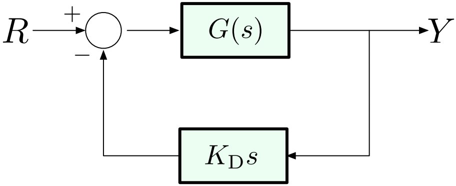
$$\dot{e}(t) \approx \frac{e(t+\delta) - e(t)}{\delta},$$

but then we must tolerate delays — must wait until time $t + \delta$ to issue a control signal meant for time t.

Disclaimer 2 about D-Feedback: Noise Amplification

Differentiators amplify noise (noise $\longrightarrow$ rapid changes in the reference).

In the lab, D-feedback is implemented differently, in the feedback path. This way, we avoid differentiating the reference, which may be rapidly changing:



Before: $\dfrac{Y}{R} = \dfrac{K_{\mathrm{D}} s G(s)}{1 + K_{\mathrm{D}} s G(s)}$

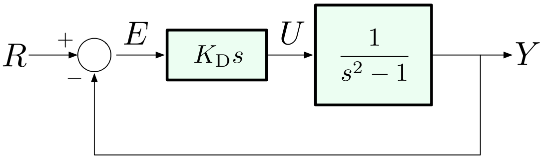
Now: $\dfrac{Y}{R} = \dfrac{G(s)}{1 + K_{\mathrm{D}} s G(s)}$

Poles: $1 + K_{\mathrm{D}} s G(s) = 0$

— same poles, but different zeros.

Now the reference signal is *smoothed out* by the plant $G(s)$ before entering the differentiator, which minimizes distortion due to noise.

Back to Analysis: Derivative Feedback



$$\frac{Y}{R} = \frac{\frac{K_D s}{s^2-1}}{1 + \frac{K_D s}{s^2-1}} = \frac{K_D s}{s^2 + K_D s - 1}$$

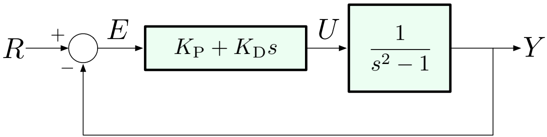
— still not good: the denominator has a negative coefficient
$\Longrightarrow$ not stable; also we have picked up a zero at the origin.

But:

- P-control affected the coefficient of s^0 (constant term)
- D-control affected the coefficient of s

— let's combine them!!

Proportional-Derivative (PD) Control



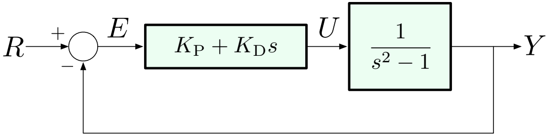
$$\frac{Y}{R} = \frac{\dfrac{K_P + K_D s}{s^2 - 1}}{1 + \dfrac{K_P + K_D s}{s^2 - 1}} = \frac{K_P + K_D s}{s^2 + K_D s + K_P - 1}$$

— now, if we set $K_D > 0$ and $K_P > 1$, then the transfer function will be stable.

Even more: by choosing K_P and K_D, we can *arbitrarily* assign coefficients of the denominator, and therefore the poles of the transfer function:

PD control gives us arbitrary pole placement!!

Proportional-Derivative (PD) Control



$$\frac{Y}{R} = \frac{K_P + K_D s}{s^2 + K_D s + K_P - 1}$$

By choosing K_P, K_D, we can achieve arbitrary pole placement!!

Also note that the addition of P-gain moves the zero:

$$K_D s + K_P = 0 \qquad \text{LHP zero at } -\frac{K_P}{K_D}$$

But what's missing? DC gain $= \left.\frac{Y}{R}\right|_{s=0} = \frac{K_P}{K_P - 1} \neq 1$

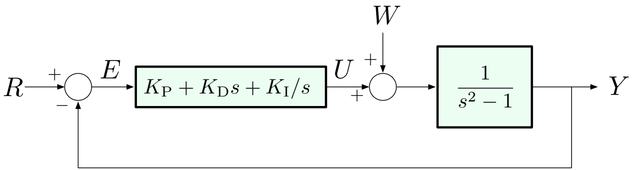
— can't have perfect tracking of constant reference.

Proportional-Integral-Derivative (PID) Control

Let us try

$$U = \left(K_\mathrm{P} + K_\mathrm{D} s + \frac{K_\mathrm{I}}{s} \right) E \qquad \text{– the classic three-term controller}$$

In fact, let's also throw in a constant disturbance:

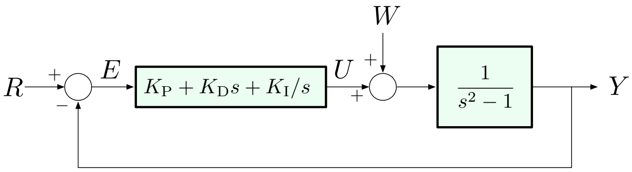


We will see that, with PID control, the goals of

- tracking a constant reference r
- rejecting a constant disturbance w

can be accomplished in one shot.

Proportional-Integral-Derivative (PID) Control



$$Y = \frac{1}{s^2-1}(U+W), \qquad U = \left(K_P + K_D s + \frac{K_I}{s}\right)(R-Y)$$

$$\text{so } Y = \frac{K_P + K_D s + \frac{K_I}{s}}{s^2-1}(R-Y) + \frac{1}{s^2-1}W$$

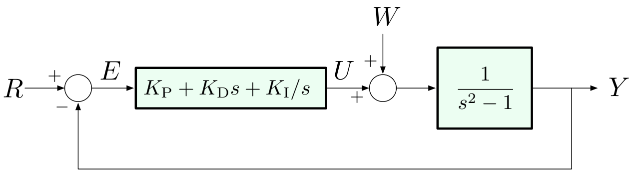
Simplify:

$$(s^2-1)Y = \left(K_P + K_D s + \frac{K_I}{s}\right)(R-Y) + W$$

$$\left(s^2 - 1 + K_P + K_D s + \frac{K_I}{s}\right)Y = \left(K_P + K_D s + \frac{K_I}{s}\right)R + W$$

$$(s^3 - s + K_P s + K_D s^2 + K_I)Y = (K_P s + K_D s^2 + K_I)R + Ws$$

Proportional-Integral-Derivative (PID) Control

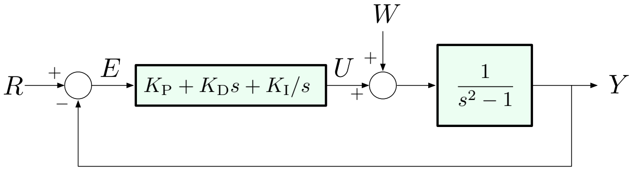


$$(s^3 - s + K_P s + K_D s^2 + K_I)Y = (K_P s + K_D s^2 + K_I)R + Ws$$

Therefore,

$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Proportional-Integral-Derivative (PID) Control

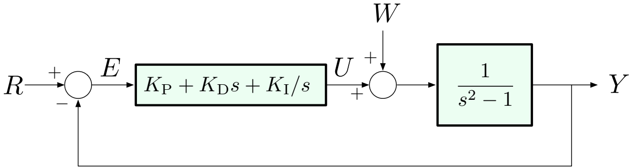


$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Stability:

- need $K_D > 0$, $K_P > 1$, $K_I > 0$ (necessary condition) and $K_D(K_P - 1) > K_I$ (Routh–Hurwitz for 3rd-order)
- can still assign coefficients arbitrarily by choosing K_P, K_I, K_D

Proportional-Integral-Derivative (PID) Control



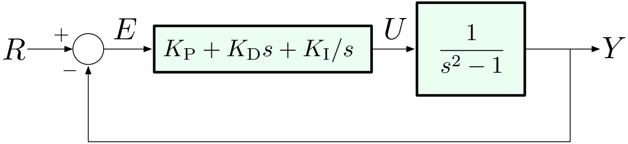
$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Reference tracking:

$$\text{DC gain}(R \to Y) = \left. \frac{K_D s^2 + K_P s + K_I}{s^3 + (K_P - 1)s + K_D s^2 + K_I} \right|_{s=0} = 1$$

— so, with the addition of I-feedback, we remove earlier limitation and achieve *perfect tracking*!

Proportional-Integral-Derivative (PID) Control



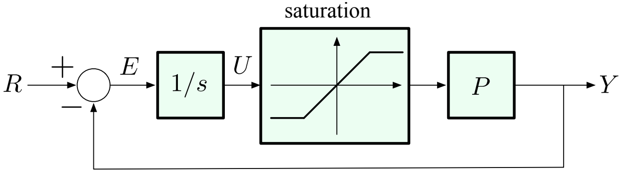
$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Disturbance rejection:

$$\text{DC gain}(W \to Y) = \left. \frac{s}{s^3 + (K_P - 1)s + K_D s^2 + K_I} \right|_{s=0} = 0$$

— so, integral gain also gives *complete attenuation* of *constant* disturbances!!

Wind-Up Phenomenon



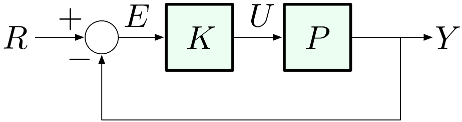
When the actuator saturates, the error continues to be integrated, resulting in large overshoot.

We say that the integrator "winds up:" the error may be small, but its integrated past history builds up.

There are various *anti-windup* schemes to deal with this practically important issue. (Essentially, we attempt to detect the onset of saturation and turn the integrator off.)

System Type

The fact that $1/s$ leads to perfect tracking of constant references and perfect rejection of constant disturbances is a special case of a more general analysis.



Consider the reference $r(t) = \frac{t^k}{k!} 1(t) \quad \longleftrightarrow \quad R(s) = \frac{1}{s^{k+1}}$

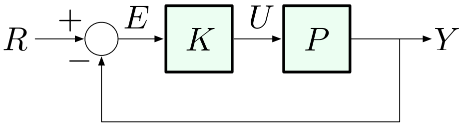
Error signal: $E = \frac{1}{1+KP} R = \frac{1}{1+KP} \frac{1}{s^{k+1}}$

FVT gives (assuming stability):

$$e(\infty) = sE(s)\Big|_{s=0} = \frac{1}{1+KP} \frac{1}{s^k}\Bigg|_{s=0}$$

— let's see how the forward gain affects tracking performance.

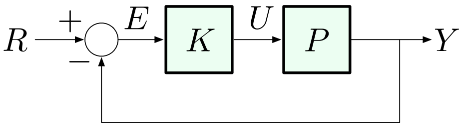
System Type



System type: the number n of poles the forward-loop transfer function KP has at the origin. It is the degree of the lowest-degree polynomial that cannot be tracked *in feedback* with zero steady-state error.

Note: the system type is calculated from the *forward-loop* transfer function, although the conclusions we will draw are about the *closed-loop system*.

System Type



$$R(s) = \frac{1}{s^{k+1}} \implies E = \frac{1}{1+KP} R = \frac{1}{1+KP} \frac{1}{s^{k+1}}$$

$$e(\infty) = sE(s)\Big|_{s=0} = \frac{1}{1+KP}\frac{1}{s^k}\Bigg|_{s=0}$$

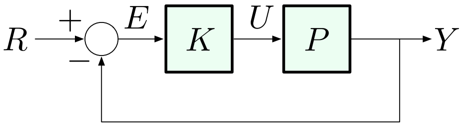
— let's see how forward gain KP affects tracking performance.

Let's suppose that KP has nth-order pole at $s = 0$: $KP = \dfrac{K_0}{s^n}$

$$sE(s) = \frac{1}{\left(1 + \frac{K_0}{s^n}\right) s^k} = \frac{s^{n-k}}{s^n + K_0}$$

— what about $sE(s)\Big|_{s=0}$?

System Type



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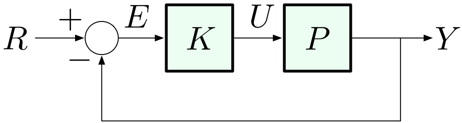
$$sE(s) = \frac{1}{\left(1 + \frac{K_0}{s^n}\right) s^k} = \frac{s^{n-k}}{s^n + K_0} \qquad \text{— what about } sE(s)\Big|_{s=0}?$$

Recall: reference $r(t)$ is a polynomial of degree k

Three cases to consider —

- $n > k$: $e(\infty) = 0$ perfect tracking
- $n = k$: $e(\infty) = \text{const} \neq 0$ imperfect tracking
- $n < k$: $e(\infty) = \infty$ no tracking

System Type: Examples



System type is the degree of the lowest-degree polynomial that cannot be tracked *in feedback* with zero steady-state error.

- Type 0: no pole at the origin. This is what we had without the I-gain: nonzero SS error to constant references.
- Type 1: a single pole at the origin. This is what we get with I-gain: can track (respectively, reject) constant references (respectively, disturbances) with zero error.
  - can check that we have a nonzero (but finite) error when tracking ramp references
- Type 2: a double pole at the origin. Can track ramp references without error, but not $t^2, t^3, \ldots$

PID Control: Summary & Further Comments

P-gain simplest to implement, but not always sufficient for stabilization

D-gain helps achieve stability, improves time response (more control over pole locations)

- arbitrary pole placement only valid for 2nd-order response; in general, we still have control over two *dominant poles*
- cannot be implemented directly, so need approximate implementation; D-gain also amplifies noise

I-gain essential for perfect steady-state tracking of constant reference and rejection of constant disturbance

- but $1/s$ is not a stable element by itself, so one must be careful: it can destabilize the system if the feedback loop is broken (integrator wind-up)