ECE 486, Spring 2018 MIDTERM #2 **Thursday, Apr 12th**

You may not open the exam book or start your exam until you are told otherwise.

Instructions:

- Read each problem carefully.
- Write legibly and in complete sentences.
- Justify *all* your answers to earn full credit.
- Cross out *anything* you do not want to be graded.
- You *must not* use books or electronics except for
  - a sheet of notes of the size up to letter (8.5×11 inch), double sided;
  - a simple calculator not capable of symbolic computing.
- You have 80 minutes to complete this exam.
- Respond to all *four* problems.

Stay calm and good luck!

Name (Print): YÜN HAN

1. (*20 points*)

   (a) (*6 points*) Consider a minimum phase transfer function $G(s)$. If a scalar gain K manages to make the local slope ≈ -45 db per decade at the crossover frequency ω_c in the Bode magnitude plot of $KG(s)$, for a unity *negative* feedback configuration, is the closed-loop system stable? Why or why not? Justify your answer.

   [*Hint*: You can assume the crossover frequency is far away from any breakpoint frequencies so that you can use Bode's Phase-Gain relationship.]

solution: Note, -2 / decade (in absolute units)

$\longleftrightarrow$ -40 dB/decade

local slope $\cong -45$ dB per decade $\Rightarrow$ local slope $= -\frac{45}{20}$

$= -2.25$. (in abs units).

By Bode's Phase-Gain relationship, the phase at crossover frequency

is about $-2.25 \times \frac{\pi}{2} = -\left(\frac{2.25}{2}\right)\pi$

$< (-1)\pi$.

i.e. PM $< 0°$.

The closed-loop system is not stable.

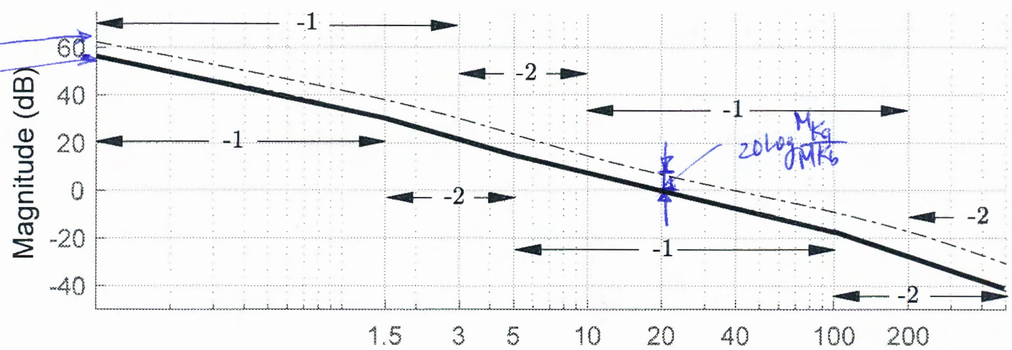
(b) (*14 points*) Consider a system forward path transfer function $G(s)$. The (asymptotic) Bode magnitude plot below shows the magnitude **before** (thick solid line) and **after** (dash dotted line) a controller $G_c(s)$ was added to the forward path (in serial connection). By this Bode plot, what is the transfer function of this controller $G_c(s)$? You can leave $G_c(s)$ in Bode form. -1 and -2 in the figure refer to local slopes. Slopes above the graphs are for the **after** and slopes below for the **before**.

after K_a

before K_b

$20\log M_{Ka} = K_a$

$20\log M_{Kb} = K_b$



solution: Before $G_c(s)$ was added, $G(s) = \dfrac{M_{Kb}}{s} \cdot \dfrac{1}{\left(\frac{1}{1.5}s+1\right)} \cdot \left(\frac{1}{5}s+1\right) \cdot \dfrac{1}{\frac{1}{100}s+1}$

After $G_c(s)$ was added, $G_c(s)G(s) = \dfrac{M_{Ka}}{s} \cdot \dfrac{1}{\frac{1}{3}s+1} \cdot \left(\frac{1}{10}s+1\right) \dfrac{1}{\frac{1}{200}s+1}$

Therefore $G_c(s) = \left(\dfrac{M_{Ka}}{M_{Kb}}\right) \dfrac{\frac{1}{1.5}s+1}{\frac{1}{3}s+1} \cdot \dfrac{1/\left(\frac{1}{5}s+1\right)}{1/\left(\frac{1}{10}s+1\right)} \cdot \dfrac{\frac{1}{100}s+1}{\frac{1}{200}s+1}$

[M_{Ka} can be calculated by $|G_c(j40)\,G(j40)| = 1$, $M_{Ka} \cong 134$.

M_{Kb} can be calculated by $|G(j20)| = 1$, $M_{Kb} \cong 67$.]

But we only need the ratio $\left(\dfrac{M_{Ka}}{M_{Kb}}\right)$.

$\dfrac{d(\log M)}{d(\log w)} = -1$, Note, by geometry, $\dfrac{\left(\log \frac{M_{Ka}}{M_{Kb}}\right)}{(\log w)\big|_{20}^{40}} = -1/\text{decade}$ is the local slope. $\Rightarrow \dfrac{M_{Ka}}{M_{Kb}} = 2$

so, $G_c(s) = 2 \cdot \dfrac{\left(\frac{1}{1.5}s+1\right)}{\left(\frac{1}{3}s+1\right)} \cdot \dfrac{\left(\frac{1}{10}s+1\right)}{\left(\frac{1}{5}s+1\right)} \cdot \dfrac{\left(\frac{1}{100}s+1\right)}{\left(\frac{1}{200}s+1\right)}$

2. (*10 points*) Consider the unity *negative* feedback loop with plant

$$G_p(s) = \frac{1}{s(s+8)}.$$

Design a Lead Compensator $G_c(s)$ such that the closed-loop system has a pair of poles at $s = -2$.

solution: ① if we can use pole cancellation, $G_c(s) = K\frac{s+8}{s+p}$. then

$$1 + G_c(s)G_p(s) = 1 + K\frac{s+8}{s+p}\cdot\frac{1}{s(s+8)}$$

$$= 1 + K\frac{1}{(s+p)s}$$

characteristic polynomial is $s^2 + ps + K$

matching $(s+2)^2 = s^2+4s+4$, we have $K=4$, $p=4$. but $p=4 < 8 = z$

rule out

② consider the case: there is no pole cancellation, the characteristic polynomial will be 3rd order.

$$1 + G_c(s)G_p(s) = 1 + K\frac{s+z}{s+p}\cdot\frac{1}{s(s+8)}$$

char. poly. $s(s+8)(s+p) + K(s+z)$

$$= s^3 + (8+p)s^2 + (8p+K) + Kz$$

desired char. poly $= (s+2)^2(s-\alpha)$ $(\alpha \in \mathbb{R})$

$$= s^3 + (4-\alpha)s^2 + (4-4\alpha)s - 4\alpha$$

matching coefficients,
$$\begin{cases} 8+p = 4-\alpha & \Rightarrow p = -4-\alpha > 0 \\ 8p+K = 4-4\alpha & \Rightarrow K = 36+4\alpha > 0 \\ Kz = -4\alpha & \Rightarrow z = \frac{-4\alpha}{36+4\alpha} > 0 \end{cases}$$

and lead compensator requires $z = \frac{-4\alpha}{36+4\alpha} < -4-\alpha = p$

$$\Rightarrow \frac{4\alpha}{36+4\alpha} > \alpha+4$$

$$\Rightarrow 4\alpha > (\alpha+4)(36+4\alpha) \quad \text{for } \alpha \in \mathbb{R}.$$

$$\Rightarrow \alpha^2 + 12\alpha + 36 < 0 \Rightarrow (\alpha+6)^2 < 0 \text{ impossible.}$$

There is no such a lead compensator

3. *(10 points)* Consider the following transfer function

$$G(s) = \frac{(s-4)^2}{(s^2+3s+2)(s-3)}.$$

Sketch the root locus of $1+KG(s)$. According to your root locus plot, is it possible to make $G(s)$ closed-loop stable using positive $K>0$?

[*Hint*: 1. At some point, if you need to solve a cubic polynomial equation $s^3-12s^2+7s+40=0$, you can use the fact that the roots are $s_1 \approx 11.0375$, $s_2 \approx 2.4449$, $s_3 \approx -1.4823$.

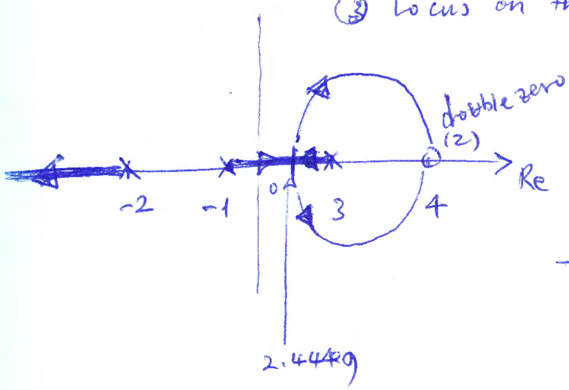
2. You do not have to justify the complex loci form a circle. As a take-home extra credit question, if you can show it actually **does not** form a circle, I will give you +3 points for this question. Take-home response is due 11:59 p.m. today.]

solution: (1) There are 3 openloop poles

2 openloop zeros

(2) Therefore one $= (3-2)$ branch of locus ends at infinity (must be real locus by symmetry)

(3) locus on the real axis. $(-\infty, -2] \cup [-1, 3]$



(4) break point can be obtained from $\frac{d}{ds}G(s) = 0$

$$\frac{d}{ds}G(s) = \frac{-[(s+1)(s+2)(s-3)]'(s-4)^2 + (s+1)(s+2)(s-3)\cdot 2(s-4)}{(\cdot)^2} = 0$$

$$\Rightarrow (s-4)\left[(s-4)(3s^2-7) - 2(s^3-7s-6)\right] = 0$$

$$\Rightarrow (s-4)(s^3-12s^2+7s+40) = 0$$

$s_1 = 4$, $\cancel{s_2 = 11.0375}$, $s_3 = 2.4449$, $\cancel{s_4 = -1.4823}$

s_1: break in point

s_2: not on the locus

s_3: break away point

s_4: not on the locus

(5) why it does not cross $j\omega$-axis. Suppose it does,

at some point it has $\begin{cases} \text{one real pole} \le -2 \\ \text{two complex poles on LHP} \end{cases} \Rightarrow$ system is stable

but for any $K>0$ the char. poly fails necessary condition for stability

$$(s^2+3s+2)(s-3) + K(s-4)^2 = s^3 - 7s - 6 + Ks^2 - 8Ks + 16K$$
$$= s^3 - \underbrace{(8K+7)}_{<0} s^2 + Ks^2 + (16K-6)$$

There is always one RHP closed-loop pole.

4. *(20 points)* Recall from our lecture, it is known that for a double integrator $G(s) = \dfrac{1}{s^2}$ as our plant, a scalar control gain K cannot give a strictly stable closed-loop system in a unity *negative* feedback configuration. Indeed, when $K > 0$, the characteristic equation

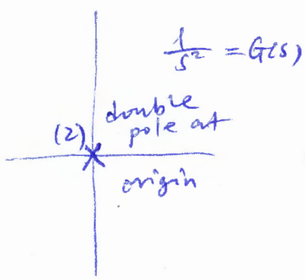
$$s^2 + K = 0$$

gives a pair of conjugate closed-loop poles on the imaginary axis; when $K < 0$, one of the closed-loop poles $s_{1,2} = \pm\sqrt{-K}$ is in the Right Half Plane (RHP). Hence in both cases the closed-loop system is not strictly stable.

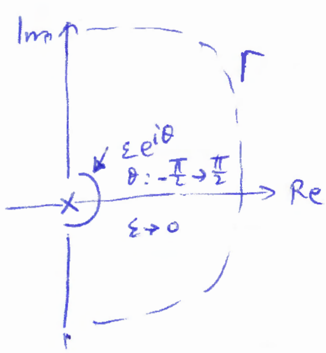
(a) *(10 points)* Let $G(s) = \dfrac{1}{s^2}$ as in the above comment, now apply Nyquist Stability Criterion to $G(s)$ to confirm that there is a one RHP closed-loop pole if $K < 0$. Justify your reasoning.

[*Hint*: You can consider a modified contour γ to enclose the entire RHP. Remember γ cannot pass origin; so the usual contour—imaginary axis union large semicircular path does not work here.]

solution:



choose modified contour to avoid double pole $s=0$

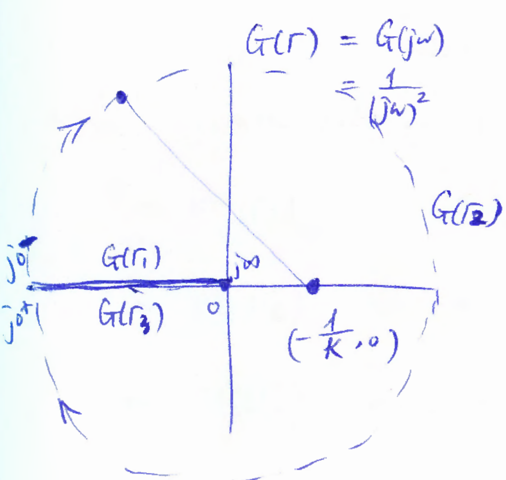


the Nyquist plot of $G(s) = \frac{1}{s^2}$ is the image curve

$G(\Gamma)$ under $\frac{1}{s^2}$

$G(\Gamma)$ has three components (excluding $G(\infty)=0$), since Γ has three components

$$\Gamma : \underbrace{(-\infty, +\varepsilon e^{i(-\frac{\pi}{2})}]}_{\Gamma_1} \cup \underbrace{[+\varepsilon e^{i(-\frac{\pi}{2})}, +\varepsilon e^{i(\frac{\pi}{2})}]}_{\substack{\text{path, not interval} \\ \Gamma_2}} \cup \underbrace{[+\varepsilon e^{i(\frac{\pi}{2})}, +\infty)}_{\Gamma_3}$$



$G(\Gamma_1)$ and $G(\Gamma_3)$ are negative real axis.

$G(\Gamma_2)$ forms a circular path with phase change

$$(-2)\cdot\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = -2\pi$$

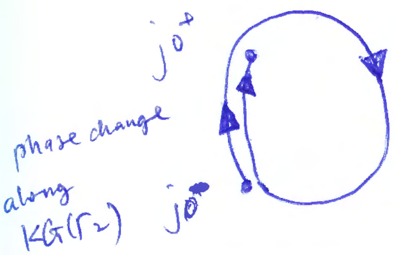
so the clockwise encirclement around $(-\frac{1}{K}, 0)$ is $1 = N$

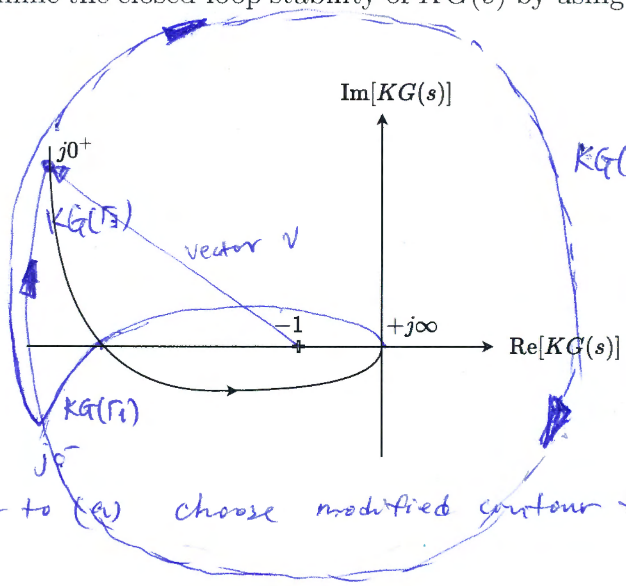
there is no open loop pole inside Γ. $P = 0 \Rightarrow Z = N + P = 1$.

(b) *(10 points)* As a follow-up, consider the following forward path transfer function

$$KG(s) = \frac{K(\tau_1 s + 1)(\tau_2 s + 1)}{s^3},$$

where $K, \tau_1, \tau_2 > 0$ all positive. Part of the Nyquist plot of $KG(j\omega)$ is already given below for $\{s = j\omega \in \mathbb{C} \mid 0^+ \leq \omega \leq +\infty\}$. With unity *negative* feedback configuration, determine the closed-loop stability of $KG(s)$ by using Nyquist Stability Criterion.





Solution: Similar to (a) choose modified contour to avoid triple pole at origin.

By the Nyquist plot of $KG(j\omega)$, we compute the encirclement about (-1)

$$\Gamma : \underbrace{(-\infty, +\varepsilon e^{i(-\frac{\pi}{2})}]}_{\Gamma_1} \cup \underbrace{[+\varepsilon e^{i(-\frac{\pi}{2})}, +\varepsilon e^{i(\frac{\pi}{2})}]}_{\Gamma_2} \cup \underbrace{[+\varepsilon e^{i(\frac{\pi}{2})}, +\infty)}_{\Gamma_3}$$

To complete the Nyquist plot, first take the conjugate of $KG(\Gamma_3)$ (given) we get the portion corresponding to $KG(\Gamma_1)$. Then $KG(\Gamma_2)$ corresponds to the large dashed circular path.

join the point $(-1, 0)$ with a point on the Nyquist plot, say this is a vector v.

on $KG(\Gamma_3)$: $\omega: 0^+ \to +\infty$, net phase change of vector v is $+270^\circ$

on $KG(\Gamma_1)$: $\omega: -\infty \to 0^-$, net phase change of vector v is $+270^\circ$

on $KG(\Gamma_2)$: $\omega: 0^- \to 0^+$ along the path $\varepsilon e^{i(-\frac{\pi}{2})}$ to $\varepsilon e^{i(\frac{\pi}{2})}$ net phase change is $(-3)\left(\frac{\pi}{2} - (-\frac{\pi}{2})\right)$
$= -540^\circ$

so the encirclement (clockwise) about $(-1, 0)$ is $0. = N$

No open loop poles $\Rightarrow P = 0$ so $Z = N + P = 0$, closed loop system is stable.