ECE 486, Spring 2018 MIDTERM #1 **Tuesday, Feb 27th**

You may not open the exam book or start your exam until you are told otherwise.

Instructions:

- Read each problem carefully.
- Write legibly and in complete sentences.
- Justify *all* your answers to earn full credit.
- Cross out *anything* you do not want to be graded.
- You *must not* use books or electronics except for
  - a sheet of notes of the size up to letter (8.5 × 11 inch), double sided;
  - a simple calculator not capable of symbolic computing.
- You have 80 minutes to complete this exam.
- Respond to all *four* problems.

Stay calm and good luck!

Name (Print): YUN HAN

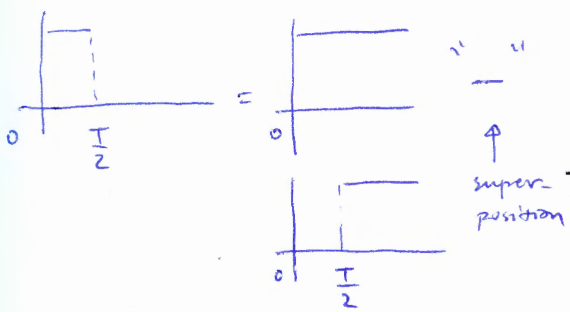
1. (*10 points*) In this problem, let the period of sine function with frequency ω be $T = \frac{2\pi}{\omega}$. You may use the following fact without proof. For a function in t with time delay c, we have

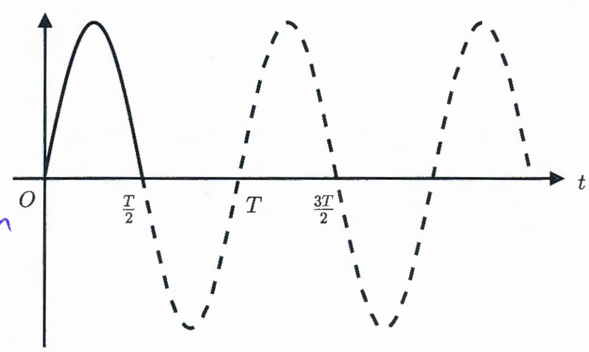
$$\mathcal{L}\{f(t-c)1(t-c)\} = \mathcal{L}\{f(t)\}e^{-cs},$$

where $1(t)$ is the unit step, $\mathcal{L}\{f(t)\}$ is the Laplace transform of $f(t)$. This function $f(t)$ is regular enough so we do not have to worry about the existence of its transform.

(a) (*4 points*) Compute the Laplace transform of $f(t) = \sin(\omega t)$, $0 \le t \le \frac{T}{2}$, where ω is the frequency, T is the period.

[*Hint*: Think of $1(t)$ as a switch, i.e., "turned on" from $t = 0$ on. Here you want to "turn off" the normal sine function from $t = \frac{T}{2}$ on.]





solution:

$$\sin(\omega t)\Big|_{0\le t\le \frac{T}{2}} = \sin(\omega t)\left(1(t) - 1(t-\tfrac{T}{2})\right)$$

$$= \sin(\omega t)\,1(t) - \sin(\omega t)\,1(t-\tfrac{T}{2})$$

$$= \sin(\omega t) - \sin\left(\omega(t-\tfrac{T}{2}) + \tfrac{\omega T}{2}\right) 1(t-\tfrac{T}{2})$$

$$= \sin(\omega t) - \sin\left(\omega(t-\tfrac{T}{2}) + \pi\right) 1(t-\tfrac{T}{2})$$

$$= \sin(\omega t) + \sin\left(\omega(t-\tfrac{T}{2})\right) 1(t-\tfrac{T}{2})$$

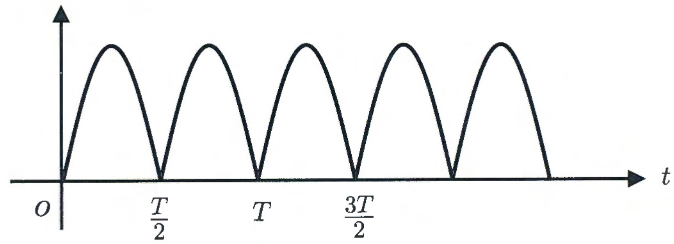
By delayed signal property above

$$\mathcal{L}\left\{\sin(\omega t)\Big|_{0\le t\le\frac{T}{2}}\right\} = \mathcal{L}\{\sin(\omega t)\} + \mathcal{L}\{\sin(\omega t)\}e^{-\frac{T}{2}s}$$

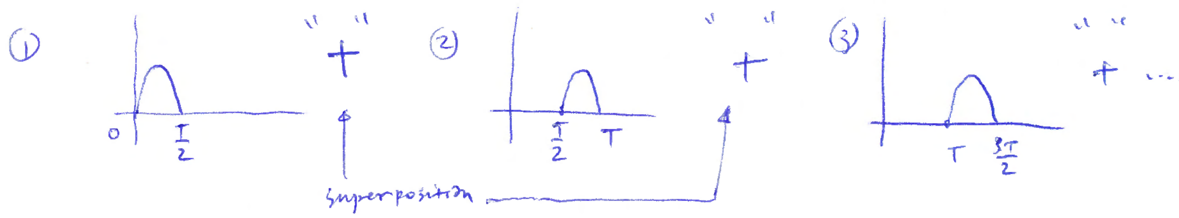
$$= \frac{\omega}{s^2+\omega^2} + \frac{\omega}{s^2+\omega^2}e^{-\frac{T}{2}s}$$

$$= \frac{\omega}{s^2+\omega^2}\left(1 + e^{-\frac{T}{2}s}\right) \quad \#$$

(b) (*6 points*) Compute the Laplace transform of $f(t) = |\sin(\omega t)|$. You may use the intuition from (a).
[*Hint*: At some point, if you need to use geometric series, you may write down infinite sum $1 + q + q^2 + \cdots = \frac{1}{1-q}$ without justification for its convergence.]



Solution: The above figure is the overlap of the following sequence



we know the transform for ① from what we did in (a)

which is $\frac{\omega}{s^2+\omega^2}\left(1 + e^{-\frac{T}{2}s}\right)$

② is ① delayed by $\frac{T}{2}$

③ is ① delayed by $\left(\frac{T}{2}\right)\cdot 2$

④ is ① delayed by $\left(\frac{T}{2}\right)\cdot 3$

⋮

so $\mathcal{L}\{|\sin(\omega t)|\} = \mathcal{L}\{\text{transform in (a)}\}\left(1 + \underbrace{e^{-\frac{T}{2}s} + e^{(-\frac{T}{2})\cdot 2s} + e^{(-\frac{T}{2})\cdot 3s} + \cdots}_{\text{geometric series}}\right)$

$$= \frac{\omega}{s^2+\omega^2} \cdot \frac{1 + e^{-\frac{T}{2}s}}{1 - e^{-\frac{T}{2}s}}$$

#

SAnity check: DC gain is 10

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$1.2^2 + 1.6^2 = 1.44 + 2.56 = 4$

2. (*15 points*) For a second order system, its *unit* step response is given by

$$y(t) = 10 - 12.5e^{-1.2t}\sin(1.6t + 53.1°),$$

where $y(t)$ is the response signal.

$\sin 53° \cong 0.8$
$\cos 53° \cong 0.6$

(a) (*9 points*) Use *two* different methods to compute the second order transfer function. [*Hint*: 53.1° is the larger acute angle of the right triangle with Pythagorean sides 3, 4 and 5.]

solution: Method #1: Denote the target transfer function as $G(s)$

$$\mathcal{L}\{y(t)\} = Y(s) = G(s)\,\mathcal{L}\{1(t)\} \Rightarrow G(s) = \frac{\mathcal{L}\{y(t)\}}{\mathcal{L}\{1(t)\}}$$

for given $y(t)$. $\mathcal{L}\{y(t)\} = \mathcal{L}\{10 - 7.5e^{-1.2t}\sin(1.6t) - 10e^{-1.2t}\cos(1.6t)\}$

$$= \frac{10}{s} - \frac{7.5 \times 1.6}{(s+1.2)^2 + 1.6^2} - \frac{10(s+1.2)}{(s+1.2)^2 + 1.6^2}$$

$$= \frac{40}{s(s^2 + 2.4s + 4)}$$

So $G(s) = \dfrac{40}{s^2 + 2.4s + 4}$ #

Method #2. $G(s)$ by defn is $\mathcal{L}\{h(t)\}$, $h(t)$ is the impulse response

but at the same time $h(t) = \dot{y}(t) = -12.5e^{-1.2t}(-1.2)\sin(1.6t+53°) + (-12.5)e^{-1.2t}\cos(1.6t+53°)\cdot 1.6$

$$= 25e^{-1.2t}\left(\cos(1.6t+53°)\,0.8 - \sin(1.6t+53°)\,0.6\right)$$

$$= 25e^{-1.2t}\sin(1.6t + 53° - 53°)$$

$$= 25e^{-1.2t}\sin(1.6t).$$

therefore $G(s) = \mathcal{L}\{25e^{-1.2t}\sin(1.6t)\}$

$$= 25 \cdot \frac{1.6}{(s+1.2)^2 + 1.6^2}$$

$$= \frac{40}{s^2 + 2.4s + 4}$$ #

(b) (*6 points*) Compute the overshoot M_p, approximate peak time t_p and settling time t_s based on the transfer function from above.

solution: By (a) $s^2 + 2.4s + 4$ is the characteristic polynomial, monic.

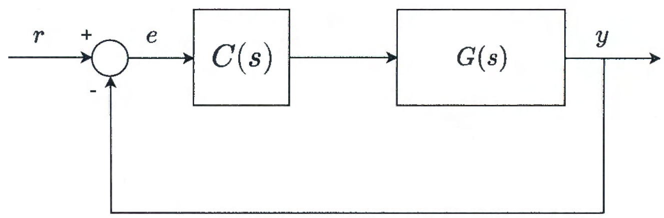
$$\begin{cases} 2\zeta\omega_n = 2.4 \\ \omega_n^2 = 4 \end{cases} \Rightarrow \begin{cases} \zeta = 0.6 \\ \omega_n = 2\,(\text{rad/s}) \end{cases}$$

$$M_p = e^{\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)} \cong 9.48\%$$ #

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\sqrt{1-\zeta^2}\cdot\omega_n} \cong 1.96\,(s)$$ #

$$t_s = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n} \cong 2.5\,(s)$$ #

3. *(20 points)* Consider the standard unity *negative* feedback loop with plant $G(s) = \dfrac{1}{s^2(s^2+s+4)}$ and controller $C(s) = \dfrac{K(4s^2+2s+1)}{s}$.



(a) *(4 points)* What is the system type? When $K = 1$, what is the DC gain of the closed loop transfer function $\frac{Y(s)}{R(s)}$?

Solution: 1° forward gain $= G(s)\,C(s) = \dfrac{K(4s^2+2s+1)}{s^3(s^2+s+4)}$

System type is 3. #

2° When $K=1$, DC gain by F.V.T. is 1, but by (b) system is not closed loop stable, so F.V.T. fails to predict the DC gain.

(b) *(8 points)* When $K = 1$, is the closed loop system stable? Why or why not? If not stable, how many unstable poles are there in the Right Half Plane?

Solution: when $K = 1$, the characteristic polynomial is

$$D(s) = s^5 + s^4 + 4s^3 + 4s^2 + 2s + 1$$

Form Routh array

limit of 1st column				
1	s^5:	1	4	2
1	s^4:	1	4	1
0^+	s^3:	ε	1	
$-\infty$	s^2:	$\dfrac{4\varepsilon-1}{\varepsilon}$	1	
1	s^1:	$1-\dfrac{\varepsilon^2}{4\varepsilon-1}$		
1	s^0:	1		

sign change (+1)

sign change (+1)

two RHP poles #

computing s^3 row

$-1\begin{vmatrix}1 & 4\\ 1 & 4\end{vmatrix} = 0$, $\varepsilon \to 0^+$

$-1\begin{vmatrix}1 & 2\\ 1 & 1\end{vmatrix} = 1$

computing s^2 row

$-\frac{1}{\varepsilon}\begin{vmatrix}1 & 4\\ \varepsilon & 1\end{vmatrix} = \dfrac{4\varepsilon-1}{\varepsilon}$

$-\frac{1}{\varepsilon}\begin{vmatrix}1 & 1\\ \varepsilon & 0\end{vmatrix} = 1$

computing s^1 row

$-\dfrac{\varepsilon}{4\varepsilon-1}\begin{vmatrix}\varepsilon & 1\\ \frac{4\varepsilon-1}{\varepsilon} & 1\end{vmatrix} = 1 - \dfrac{\varepsilon^2}{4\varepsilon-1}$

pass to the limit $\varepsilon \to 0$ first column

(c) *(8 points)* For what range of values of K is this closed loop system stable? [*Hint*: At some point, if you need to solve the quadratic equation $32x^2 - 47x + 16 = 0$, you can use the fact that its roots are $x_1 \approx 0.536$ and $x_2 \approx 0.933$.]

solution: when K is a variable, the characteristic polynomial is

$$D(s) = s^5 + s^4 + 4s^3 + (4K)s^2 + (2K)s + K$$

Form Routh array:

obtain range on the go

$$\begin{array}{lll}
 & s^5: & 1 \quad 4 \quad 2K \\
 & s^4: & 1 \quad 4K \quad K \\
\Rightarrow K<1, & s^3: & \underbrace{4(1-K)}_{>0} \quad K \\
\Rightarrow 0<K<\frac{15}{16} & s^2: & \frac{(15-16K)K^{>0}}{4(1-K)_{>0}} \quad K \\
\Rightarrow 0.536<K<0.933 & s^1: & \frac{-32K^2+47K-16^{>0}}{15-16K_{>0}} \\
\Rightarrow K>0 & s^0: & K
\end{array}$$

computing s^3 row

$$-1\begin{vmatrix} 1 & 4 \\ 1 & 4K \end{vmatrix} = 4-4K$$

$$-1\begin{vmatrix} 1 & 2K \\ 1 & K \end{vmatrix} = K$$

computing s^2 row

$$-\frac{1}{4(1-K)}\begin{vmatrix} 1 & 4K \\ 4(1-K) & K \end{vmatrix} = \frac{(15-16K)K}{4(1-K)}$$

computing s^1 row

$$-\frac{4(1-K)}{(15-16K)K}K\left(4-4K-\frac{(15-16K)K}{4(1-K)}\right)$$

$$= \frac{(15-16K)K - 4^2(1-K)^2}{15-16K}$$

$$= \frac{-32K^2+47K-16}{15-16K}$$

take the intersection of the above ranges

$$0.536 < K < 0.933$$ #

this is not about testing stability !!

4. *(15 points)* Recall from Lecture 9, a *unity* negative feedback loop with an unstable plant $G_p(s) = \frac{1}{s^2-1}$ and PD or PID controller was given. In class, we applied PD controller $G_c(s) = K_P + K_D s$ and PID controller $G_c(s) = K_P + K_D s + \frac{K_I}{s}$ respectively and we argued at the time that in both cases we had *arbitrary* pole placement.

   (a) *(7 points)* In the case of PD control, if we restrict $K_D = K_P$, do we still have arbitrary pole placement? If not, what do we have instead?
   Use the given plant $G_p(s) = \frac{1}{s^2-1}$ to carry out similar calculations like what we did in class to justify your claim.

Solution: No. poles will be restricted to root locus. Indeed

Characteristic polynomial will be determined by

$$1 + (K_p + K_D s)\, G_p(s)$$

$$= 1 + K \underbrace{(1+s)\, G_p(s)}_{:= L(s)} \quad \text{by } K = K_p = K_D \quad \text{when } G_p(s) = \frac{1}{s^2-1}.$$

$$L(s) = \frac{1+s}{s^2-1}$$

varying K, we will get a curve which is ~~the~~ root locus.

   (b) *(8 points)* Similarly, in the case of PID control, if we restrict $K_I = K_D = K_P$, do we still have arbitrary pole placement? What do we have instead in this case? Justify your claim using the given plant $G_p(s) = \frac{1}{s^2-1}$ and relevant calculations to illustrate your point.

Solution: No. for the same reason as in (a)

we will get root locus instead, which is a curve

$$1 + \left(K_p + K_D s + \frac{K_I}{s}\right) G_p(s)$$

$$= 1 + K \left(1 + s + \frac{1}{s}\right) G_p(s)$$

$$= 1 + K \underbrace{\left(\frac{s^2+s+1}{s} \; \frac{1}{s^2-1}\right)}_{:= L(s)}$$