ECE 486 Control Systems

Spring 2018

Midterm #1 Information

Issued: February 20, 2018
Updated: February 24, 2018

- This document is an info sheet about the *first* exam of ECE 486, Spring 2018.
- Please read the following information carefully and start/continue studying the *first* exam.

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- **When and where is the exam taking place?**

  The first midterm will be held on Tuesday, Feb 27, in class from 9:30 a.m. — 10:45 a.m. There is *no conflict exam* offered at any other time.

  For those who cannot make it for legitimate reasons, for each exam that is skipped, their final exam will be reweighted with an additional 15% of the course grade. For example, skipping this exam will make the final count as 40%.

- **What topics will be covered?**

  Everything covered in Lecture 1 through 9 is a fair game. That is everything up to PID control (including PID control); see lecture matrix for details.

  `https://courses.engr.illinois.edu/ece486/sp2018/lectures/`

  Here is a list of specific topics:

  – Models of simple mechanical and electrical systems; State-space equations; Linearization

  – Impulse response and convolution integral; Transfer function, frequency response; Computing transfer functions using Laplace transform, partial fractions method, final-value theorem and DC gain, real poles and transient response

  – All-integrator diagrams; Transfer functions for basic block diagrams (series, parallel, and feedback connections) and block diagram reduction

  – Prototype 2nd order response, effect of complex poles; Time-domain specifications (rise time, settling time, overshoot) and their interpretation in the s-plane; Effect of zeros; Stability and Routh-Hurwitz criterion

  – Open-loop vs. closed-loop feedback control: reference tracking and disturbance rejection, sensitivity to parameter variations, time response; Basics of PID control, system type

- **What to bring during the exam?**

  The exam is closed-book, closed-notes. You may bring one sheet (double-sided, letter size 8.5×11 inch) of notes with any necessary formulas. A simple calculator without symbolic computation is allowed.

- **Any tips for studying the exam?**

  The primary goal of the exam is to test your understanding of the main concepts, not memorization or computational skills. Make sure you can follow all the lecture material, readings, and homework problems and solutions. On the next page, an exam from a past semester is given as a sample. An outline of solutions to this sample exam ~~will be posted~~ is posted alongside the sample exam ~~on the course website two days before our exam~~.

  Disclaimer: The exam this semester will be significantly different in style and content from that older one.

- **Is there any extra office hours?**

  There ~~might be~~ is a one-hour session of extra office hours on Monday, February 26. ~~Details to be announced.~~ Room confirmed, 4034 ECEB (not 3034 for normal office hours), Monday 4 p.m. — 5 p.m., February 26.

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1. **Problem 1**: Consider the linear system given by the following differential equation

$$y^{(4)} + 3y^{(3)} + 2\ddot{y} + 3\dot{y} + 2y = \dot{u} - u,$$

where u is the input and y the output.

Is this system stable? Show your work to justify your claim. *Note*: $y^{(4)}$ is the fourth derivative of y.

Solution: Apply Laplace transform to the given differential equation with zero initial conditions,

$$(s^4 + 3s^3 + 2s^2 + 3s + 2)Y(s) = (s-1)U(s).$$

The system transfer function is therefore

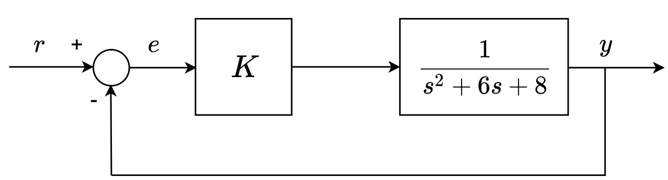
$$\frac{Y(s)}{U(s)} = \frac{s-1}{s^4 + 3s^3 + 2s^2 + 3s + 2}.$$

All the coefficients of the characteristic polynomial $s^4 + 3s^3 + 2s^2 + 3s + 2$ are positive, necessary condition for stability is met. We need to further check stability with Routh test. (So, if necessary condition is *not* met, there is no need to test Routh-Hurwitz.) Form Routh array

$$\begin{array}{lccc} s^4: & 1 & 2 & 2 \\ s^3: & 3 & 3 & 0 \\ s^2: & 1 & 2 & \\ s^1: & -3 & 0 & \\ s^0: & 2 & & \end{array}$$

Not all the entries in the first column are positive ($-3 < 0$). Therefore the system is not stable.

2. **Problem 2**: Consider the unity *negative* feedback loop with plant $G(s) = \frac{1}{s^2+6s+8}$ and proportional controller $C(s) = K$. Assume all initial conditions are zero here.



(a) What is the transfer function from r to y?

Solution: $\dfrac{Y}{R}(s) = \dfrac{K}{s^2 + 6s + (K+8)}.$

(b) For what range of values of K is this closed loop system stable?

Solution: To the characteristic polynomial $s^2 + 6s + (K + 8)$, we apply the Routh-Hurwitz criterion for second order transfer function, i.e., it is necessary and sufficient that all coefficients are positive. We need $K > -8$.

(c) Suppose we want the closed loop system to be underdamped with peak time $t_p < 2\pi$ seconds. Sketch the possible pole locations in the complex plane. Mark indicative regions or points if they are helpful.

Solution: Peak time $t_p = \frac{\pi}{\omega_d} \leq 2\pi$ implies $\omega_d \geq \frac{1}{2}$. But ω_d is the damped natural frequency when system is underdamped. Sketch the closed loop poles on the complex plane

$$s_{1,2} = -\sigma \pm j\omega_d$$

with negative real part and imaginary part with absolute value ≥ 0.5.

(d) For what range of values of K is the closed loop system underdamped with peak time $t_p < 2\pi$ seconds?

Solution: From (c), the closed loop poles are

$$s_{1,2} = -3 \pm \sqrt{1-K}.$$

When the system is underdamped, $1 - K < 0$, i.e., $K > 1$. The closed loop poles further become

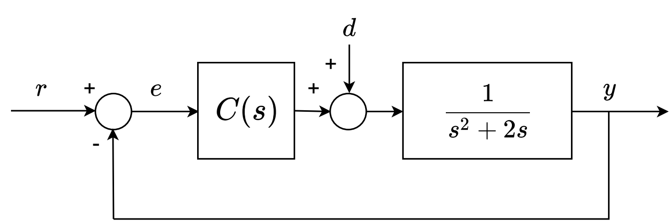
$$s_{1,2} = -3 \pm j\sqrt{K-1}.$$

By (c), the absolute value of the imaginary part ≥ 0.5. So

$$\sqrt{K-1} \geq 0.5.$$

We get $K \geq \frac{5}{4}$.

3. **Problem 3**: Consider the unity *negative* feedback loop with reference r, output y, tracking error e, input disturbance d and plant $P(s) = \frac{1}{s^2+2s}$. Assume all initial conditions are zero here.



(a) Suppose we want to design a controller $C(s)$ so that the steady state tracking error due to square input $r(t) = t^2$, $t \geq 0$ is at most 0.5 ($d = 0$, no disturbance). With only two possible choices, P and PID controllers, which controller would you use? And why?

Solution: To track quadratic reference input t^2 with error up to a nonzero constant, we need the system type to be 2. The plant $P(s)$ itself already has 1 pole at origin, we need one more from the controller. Therefore we choose PID control.

(b) Suppose we choose the PID controller $C(s) = \frac{K_{\mathrm{D}}s^2+K_{\mathrm{P}}s+K_{\mathrm{I}}}{s}$. For square input $r(t) = t^2$, $t \geq 0$ without disturbance $d = 0$, derive the expression for the steady state error in terms of gains K_{P}, K_{D} and/or K_{I}. What conditions (if any) must the gains satisfy in order to achieve a steady state error of less than 0.5?

Solution: The transfer function from reference to error is

$$\begin{aligned}\frac{E}{R}(s) &= \frac{1}{1+P(s)C(s)} \\ &= \frac{1}{1+\frac{1}{s^2+2s}\frac{K_{\mathrm{D}}s^2+K_{\mathrm{P}}s+K_{\mathrm{I}}}{s}} \\ &= \frac{s^3+2s^2}{s^3+(K_{\mathrm{D}}+2)s^2+K_{\mathrm{P}}s+K_{\mathrm{I}}}.\end{aligned}$$

Apply Routh-Hurwitz criterion for third order transfer function, all the coefficients of the characteristic polynomial shall be positive

$$K_{\mathrm{D}}+2>0,\ K_{\mathrm{P}}>0,\ K_{\mathrm{I}}>0,$$

and $(K_{\mathrm{D}}+2)K_{\mathrm{P}} > K_{\mathrm{I}}$.

After we guarantee stability, we further apply the Final Value Theorem

$$\begin{aligned}e_{\mathrm{ss}} &= e(\infty) \\ &= \lim_{s\to 0} sE(s) \\ &= \lim_{s\to 0} s\left[\frac{E}{R}(s)\right]R(s) \\ &= \lim_{s\to 0} s\cdot\frac{s^3+2s^2}{s^3+(K_{\mathrm{D}}+2)s^2+K_{\mathrm{P}}s+K_{\mathrm{I}}}\cdot\mathcal{L}\{t^2\} \\ &= \lim_{s\to 0} s\cdot\frac{s^3+2s^2}{s^3+(K_{\mathrm{D}}+2)s^2+K_{\mathrm{P}}s+K_{\mathrm{I}}}\cdot\frac{2}{s^3} \\ &= \frac{4}{K_{\mathrm{I}}}.\end{aligned}$$

Steady state error $e_{\mathrm{ss}} < 0.5$ implies $K_{\mathrm{I}} > 8$.

In conclusion, $K_{\mathrm{P}} > 0$, $K_{\mathrm{D}} > -2$, $K_{\mathrm{I}} > 8$ and $(K_{\mathrm{D}}+2)K_{\mathrm{P}} > K_{\mathrm{I}}$.

(c) With the controller $C(s)$ as in (b), suppose we want the closed loop poles to be at $s = -1, -2, -a$, where a is some positive real number. For $r(t) = t^2$, $t \geq 0$ and $d = 0$, what is the minimum (or lower limit if minimum cannot be achieved) value of a in order to achieve a steady state error of less than 0.5?

Solution: By (b), we try to associate a with K_{I}. Note K_{I} is the constant term of the characteristic polynomial in (b). If the closed loop poles are $s = -1, -2, -a$, the resulting characteristic polynomial is

$$\begin{aligned}&(s+1)(s+2)(s+a) \\ &= s^3+\cdots+2a. \\ &[= s^3+(a+3)s^2+(3a+2)s+2a] \qquad \text{(This is not necessary)}\end{aligned}$$

Matching the characteristic polynomial from (b), we have $K_{\mathrm{I}} = 2a$. Further from (b), $K_{\mathrm{I}} > 8$. Therefore $a > 4$.

(d) With the controller $C(s)$ as in (b), compute the transfer function from d to y ($r = 0$). Leave the answer in terms of K_{P}, K_{D} and/or K_{I}. (Do *not* substitute any numerical values.)

Solution: Reshape the loop in the figure if necessary, the forward gain is $P(s) = \frac{1}{s^2+2s}$ and the loop gain is $P(s)C(s)$ with negative feedback. Apply our formula $\dfrac{\text{forward gain}}{1+\text{ loop gain}}$, then

$$\begin{aligned}\frac{Y}{D}(s) &= \frac{P(s)}{1+P(s)C(s)} \\ &= \frac{\frac{1}{s^2+2s}}{1+\frac{1}{s^2+2s}\frac{K_{\mathrm{D}}s^2+K_{\mathrm{P}}s+K_{\mathrm{I}}}{s}} \\ &= \frac{s}{s^3+(K_{\mathrm{D}}+2)s^2+K_{\mathrm{P}}s+K_{\mathrm{I}}}.\end{aligned}$$

(e) Assuming the closed loop system is stable, what is the steady state response due to a unit step disturbance input, i.e., $d(t) = 1$, $t \geq 0$ ($r = 0$)? Is this control system good at rejecting step disturbances?

Solution: By the transfer function from disturbance to output in (d), the system type is 1. For a 0-th degree polynomial disturbance input $d(t) = 1$, the disturbance response is 0.

Alternatively, apply FVT to step disturbance response based on the transfer function in (d).

The control system is good at rejecting step disturbance inputs.