ECE 486 Control Systems

Spring 2018

Final Information

Issued: April 30, 2018
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- This document is an info sheet about the *final* exam of ECE 486, Spring 2018.
- Please read the following information carefully and start/continue studying the *final* exam.

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- **When and where is the exam taking place?**

  The final exam will be held on Friday, May 11 in 3017 ECEB from 8 a.m. — 11 a.m. There is *no conflict exam* offered at any other time.

- **What topics will be covered?**

  The final exam is *comprehensive*. Everything covered in Lecture 1 (`lec1.html`) through 23 (`lec23.html`) is a fair game. That is everything from Day 1 to the last lecture; see lecture matrix for details.

  `https://courses.engr.illinois.edu/ece486/sp2018/lectures/`

  Here is a list of specific topics:

  - All topics listed in midterm 1 and midterm 2 information sheets
  - State-space models and associated transfer functions; Linear (coordinate) transformations; Canonical Forms
  - Controllability; Pole placement by full-state feedback
  - Observability; Observer design
  - Combining full-state feedback and observer; Dynamic output feedback

- **What to bring during the exam?**

  The exam is closed-book, closed-notes. You may bring three sheets (double-sided, letter size 8.5×11 inch) of notes with any necessary formulas. A simple calculator without symbolic computation is allowed.

- **Any tips for studying the exam?**

  The primary goal of the exam is to test your understanding of the main concepts, not memorization or computational skills. Make sure you can follow all the lecture material, readings, and homework problems and solutions. On the next page, an exam from a past semester is given as a sample. An outline of solutions to this sample exam is posted alongside the sample exam.

Disclaimer: The exam this semester will be significantly different in style and content from that older one.

- **Is there any extra office hours?**

  No office hours after Reading Day. However there is a two-hour session of extra office hours in 4034 ECEB (not 3034 for normal office hours) on Thursday 1 p.m. — 3 p.m., May 10.

I wish you good luck on your final exams and a successful future! –Yün

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1. **Problem 1**:

(a) Consider the single-input linear system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u$ where

$$\boldsymbol{A} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$

in which $\lambda \in \mathbb{R}$ a real number.

Show the system is not completely controllable, i.e., there is no matrix $\boldsymbol{B}$ such that $(\boldsymbol{A}, \boldsymbol{B})$ is completely controllable.

(b) Notations as above but $\boldsymbol{A}$ changes to

$$\boldsymbol{A} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix},$$

where $\lambda_1 \neq \lambda_2$. Is it still true that $(\boldsymbol{A}, \boldsymbol{B})$ is not completely controllable for any matrix $\boldsymbol{B}$?

Solution:

(a) We notice $\boldsymbol{B}$ is a column vector so we can write

$$\boldsymbol{B} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

Then we compute the controllability matrix $\mathcal{C}(\boldsymbol{A}, \boldsymbol{B})$

$$\begin{aligned} \mathcal{C}(\boldsymbol{A}, \boldsymbol{B}) &= \left(\begin{array}{c|c|c} \boldsymbol{B} & \boldsymbol{A}\boldsymbol{B} & \boldsymbol{A}^2\boldsymbol{B} \end{array} \right) \\ &= \begin{pmatrix} b_1 & \lambda b_1 & \lambda^2 b_1 \\ b_2 & \lambda b_2 + b_3 & \lambda^2 b_2 + 2\lambda b_3 \\ b_3 & \lambda b_3 & \lambda^2 b_3 \end{pmatrix}. \end{aligned}$$

- If $b_1 = 0$, then the first row of $\mathcal{C}(\boldsymbol{A}, \boldsymbol{B})$ is a zero row, resulting in a singular controllability matrix.
- If $b_1 \neq 0$, the third row and the first row differ by a ratio $\frac{b_3}{b_1}$, i.e., they are linearly dependent; the controllability matrix is also singular.

Therefore the system is never completely controllable.

(b) With the new $\boldsymbol{A}$, the controllability matrix $\mathcal{C}(\boldsymbol{A}, \boldsymbol{B})$ becomes

$$\begin{aligned} \mathcal{C}(\boldsymbol{A}, \boldsymbol{B}) &= \left(\begin{array}{c|c|c} \boldsymbol{B} & \boldsymbol{A}\boldsymbol{B} & \boldsymbol{A}^2\boldsymbol{B} \end{array} \right) \\ &= \begin{pmatrix} b_1 & \lambda_1 b_1 & \lambda_1^2 b_1 \\ b_2 & \lambda_2 b_2 + b_3 & \lambda_2^2 b_2 + 2\lambda_2 b_3 \\ b_3 & \lambda_2 b_3 & \lambda_2^2 b_3 \end{pmatrix}. \end{aligned}$$

If we wish to find such a $\boldsymbol{B}$ that makes the controllability matrix non singular, we restrict to $b_1, b_3 \neq 0$. By elementary row operations, $\mathcal{C}(\boldsymbol{A}, \boldsymbol{B})$ can be reduced to

$$\begin{aligned}\mathcal{C}(\boldsymbol{A}, \boldsymbol{B}) &\rightsquigarrow \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 0 & \lambda_2 - \lambda_1 & \lambda_2^2 - \lambda_1^2 \\ b_2 & \lambda_2 b_2 + b_3 & \lambda_2^2 b_2 + 2\lambda_2 b_3 \end{pmatrix} \\ &\rightsquigarrow \begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 0 & 1 & \lambda_1 + \lambda_2 \\ 0 & 0 & (\lambda_2 - \lambda_1) b_3 \end{pmatrix}.\end{aligned}$$

We see that when b_1, b_3 are nonzero and $\lambda_1 \neq \lambda_2$, the controllability matrix is invertible. The system is completely controllable.

2. **Problem 2**: If the linear system $(\boldsymbol{A}, \boldsymbol{b})$ is completely controllable, it is always possible to find a $\boldsymbol{c}$ such that $(\boldsymbol{A}, \boldsymbol{c})$ is completely observable? Prove your claim.

   Solution: Yes, it is always possible. To prove the claim, we use the controllability of $(\boldsymbol{A}, \boldsymbol{b})$ to find a linear transformation matrix $\boldsymbol{T}$ such that $(\boldsymbol{A}, \boldsymbol{b})$ will be converted to $(\bar{\boldsymbol{A}}, \bar{\boldsymbol{b}})$ in CCF, where $\bar{\boldsymbol{A}}$ and $\bar{\boldsymbol{b}}$ are in the form

$$\bar{\boldsymbol{A}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & a_{n-2} & \cdots & -a_1 \end{pmatrix}$$

$$\bar{\boldsymbol{b}} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

   Now if we choose $\bar{\boldsymbol{c}} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$, we notice that the observability matrix $\mathcal{O}(\bar{\boldsymbol{A}}, \bar{\boldsymbol{c}}) = \boldsymbol{I}_n$. This choice of $\bar{\boldsymbol{c}}$ gives us a non singular observability matrix, making $(\bar{\boldsymbol{A}}, \bar{\boldsymbol{c}})$ completely observable.

   Transforming back to the original coordinate, we get

$$\begin{aligned}\boldsymbol{c} &= \bar{\boldsymbol{c}}\boldsymbol{T} \\ &= \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \boldsymbol{T},\end{aligned}$$

   where $\boldsymbol{T}$ is the transformation matrix such that $\bar{\boldsymbol{A}} = \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1}$. Then $(\boldsymbol{A}, \boldsymbol{c})$ is also completely observable since observability is preserved under linear transformation.

3. **Problem 3**: Consider the transfer function $G(s) = \dfrac{s+1}{s^2 + 2s + 1}$.

   (a) Find a second order state-space realization in Controllable Canonical Form for this transfer function $G(s)$. Check if your realization is completely observable.

(b) Find a second order state-space realization in Observable Canonical Form for this transfer function $G(s)$. Check if your realization is completely controllable.

(c) Is it possible to find a second order realization that is both completely controllable and completely observable?

Solution:

(a) One Controllable Canonical Form is given by

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u,$$
$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} \boldsymbol{x}.$$

The observability matrix associated with the above representation is

$$\begin{aligned} \mathcal{O}(\boldsymbol{A}, \boldsymbol{C}) &= \begin{pmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \end{aligned}$$

which is singular. Hence this state-space model is not completely observable.

(b) One Observable Canonical Form is given by

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u,$$
$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} \boldsymbol{x}.$$

The controllability matrix associated with the above representation is

$$\begin{aligned} \mathcal{C}(\boldsymbol{A}, \boldsymbol{B}) &= \begin{pmatrix} \boldsymbol{B} & \boldsymbol{AB} \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \end{aligned}$$

which is singular. Hence this state-space model is not completely controllable.

(c) Suppose there is a second order realization which is both completely controllable and completely observable, then this second order realization is the minimal realization of the transfer function $G(s)$. But obviously there is pole-zero cancellation. Actually $G(s) = \dfrac{1}{s+1}$.

The minimal realization of $G(s) = \dfrac{1}{s+1}$ is first order given by

$$\dot{x} = -x + u,$$
$$y = x.$$

4. **Problem 4**: Consider a linear system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u},$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$$

together with an observer

$$\dot{\hat{\boldsymbol{x}}} = (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\hat{\boldsymbol{x}} + \boldsymbol{L}\boldsymbol{y} + \boldsymbol{B}\boldsymbol{u}$$

and full-state feedback control

$$\boldsymbol{u} = -\boldsymbol{K}\hat{\boldsymbol{x}}.$$

Find the transfer function from $Y(s)$ to $U(s)$.

Solution: By control law $\boldsymbol{u} = -\boldsymbol{K}\hat{\boldsymbol{x}}$, if we can write $\hat{X}(s)$ in terms of $Y(s)$ then we win. This was done in `lec23.html`.

$$\begin{aligned} s\hat{X} &= (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\hat{X} + \boldsymbol{L}Y + \boldsymbol{B}U \\ \Longrightarrow (s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{L}\boldsymbol{C})\hat{X} &= \boldsymbol{L}Y + \boldsymbol{B}(-\boldsymbol{K}\hat{X}) \\ \Longrightarrow \hat{X} &= (s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{L}\boldsymbol{C} + \boldsymbol{B}\boldsymbol{K})^{-1}\boldsymbol{L}Y. \end{aligned}$$

Therefore the transfer function from $Y(s)$ to $U(s)$ is

$$\frac{U(s)}{Y(s)} = -\boldsymbol{K}(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{L}\boldsymbol{C} + \boldsymbol{B}\boldsymbol{K})^{-1}\boldsymbol{L}.$$

5. **Problem 5**: Consider the following linear system

$$\dot{\boldsymbol{x}} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \boldsymbol{u},$$
$$\boldsymbol{y} = \begin{pmatrix} 1 & 0 \end{pmatrix} \boldsymbol{x}.$$

   (a) Compute the open-loop transfer function based on $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C})$ given above.

   (b) Is the open-loop system stable?

   (c) Is the system completely controllable? Is the system completely observable?

   (d) Design an observer to place the observer poles at $(-2, -2)$.

   (e) Design a full-state feedback controller to place the closed-loop poles at $(-1, -1)$.

   Solution:

   (a) Let $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\boldsymbol{C} = \begin{pmatrix} 1 & 0 \end{pmatrix}$. Then the open-loop transfer function is given by

$$\begin{aligned} G(s) &= \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s & -1 \\ -1 & s+1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \frac{1}{s^2 + s - 1}. \end{aligned}$$

(b) The constant term of the characteristic polynomial is negative. The open-loop transfer function is not stable.

(c) The controllability matrix is non singular.

$$\begin{aligned}\mathcal{C}(\boldsymbol{A},\boldsymbol{B}) &= \begin{pmatrix}\boldsymbol{B} & \boldsymbol{AB}\end{pmatrix}\\ &= \begin{pmatrix}0 & 1\\ 1 & -1\end{pmatrix}.\end{aligned}$$

The observability matrix is non singular.

$$\begin{aligned}\mathcal{O}(\boldsymbol{A},\boldsymbol{C}) &= \begin{pmatrix}\boldsymbol{C}\\ \boldsymbol{CA}\end{pmatrix}\\ &= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.\end{aligned}$$

Therefore, the given state-space representation is both completely controllable and observable.

(d) Let output injection matrix $\boldsymbol{L} = \begin{pmatrix}\ell_1\\ \ell_2\end{pmatrix}$. The desired characteristic polynomial given by observer poles is $(s+2)^2 = s^2 + 4s + 4$. Matching the coefficients,

$$\begin{aligned}\det(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{LC}) &= \det\left(s\boldsymbol{I} - \begin{pmatrix}-\ell_1 & 1\\ 1-\ell_2 & -1\end{pmatrix}\right)\\ &= s^2 + (\ell_1+1)s + (\ell_1+\ell_2-1)\\ \implies \ell_1 = 3,\ \ell_2 = 2.\end{aligned}$$

Then the observer is given by

$$\dot{\hat{\boldsymbol{x}}} = (\boldsymbol{A} - \boldsymbol{LC})\hat{\boldsymbol{x}} + \boldsymbol{Ly} + \boldsymbol{Bu}.$$

(e) Let full-state feedback matrix $\boldsymbol{K} = \begin{pmatrix}k_1 & k_2\end{pmatrix}$. The desired characteristic polynomial given by closed-loop poles is $(s+1)^2 = s^2+2s+1$. Matching the coefficients,

$$\begin{aligned}\det(s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{BK}) &= \det\left(s\boldsymbol{I} - \begin{pmatrix}0 & 1\\ 1-k_1 & -1-k_2\end{pmatrix}\right)\\ &= s^2 + (k_2+1)s + (k_1-1)\\ \implies k_1 = 2,\ k_2 = 1.\end{aligned}$$

Then the full-state feedback control is given by

$$\boldsymbol{u} = -\boldsymbol{K}\hat{\boldsymbol{x}}.$$

6. **Problem 6**: Consider the following system

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}-x_1^2+1\\ -x_2\end{pmatrix}.$$

(a) Find the equilibrium points of the system.

(b) Check the equilibrium points found in the previous question whether they are

i. Stable

ii. Unstable

Solution:

(a) Let the right hand side be zero vector and solve for the equilibrium points.

$$\begin{pmatrix} -x_1^2 + 1 \\ -x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\implies \boldsymbol{x}_e = \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}.$$

There are two equilibrium points.

(b) The Jacobian of the right hand side is

$$\frac{\partial}{\partial \boldsymbol{x}} \text{RHS} = \begin{pmatrix} -2x_1 & 0 \\ 0 & -1 \end{pmatrix}.$$

i. Evaluating it at the first equilibrium point, we get

$$\left.\begin{pmatrix} -2x_1 & 0 \\ 0 & -1 \end{pmatrix}\right|_{\boldsymbol{x}_e = \begin{pmatrix} -1 \\ 0 \end{pmatrix}} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix},$$

where one of the eigenvalues is positive. This equilibrium point is not stable.

ii. Evaluating it at the second equilibrium point, we get

$$\left.\begin{pmatrix} -2x_1 & 0 \\ 0 & -1 \end{pmatrix}\right|_{\boldsymbol{x}_e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix},$$

where both eigenvalues are in LHP. This equilibrium point is stable.

7. **Problem 7**: Consider the differential equation

$$\ddot{y} + (4 + y^3)\dot{y} + 2y(1 + 3y^2) = 2u + 4\dot{u}$$

with input u and output y.

(a) Linearize the differential equation at $y = 0$.

(b) Find a state-space realization in Controllable Canonical Form for the resulting linear system from the previous question.

Solution:

(a) Use Taylor expansion against y at 0 for both $4+y^3$ and $1+3y^2$. We have $4+y^3 \approx 4$ and $1+3y^2 \approx 1$. Then the differential equation becomes

$$\ddot{y} + 4\dot{y} + 2y = 4\dot{u} + 2u.$$

(b) The transfer function according to the linearized differential equation above is

$$\frac{Y}{U}(s) = \frac{4s+2}{s^2+4s+2}.$$

One Controllable Canonical Form is given by

$$\begin{aligned} \dot{\boldsymbol{x}} &= \begin{pmatrix} 0 & 1 \\ -2 & -4 \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \\ y &= \begin{pmatrix} 2 & 4 \end{pmatrix} \boldsymbol{x}. \end{aligned}$$