Plan of the Lecture

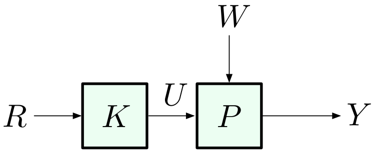
- **Review:** stability; Routh–Hurwitz criterion
- **Today's topic:** basic properties and benefits of feedback control

Goal: understand the difference between open-loop and closed-loop (feedback) control; examine the benefits of feedback: reference tracking and disturbance rejection; reduction of sensitivity to parameter variations; improvement of time response.

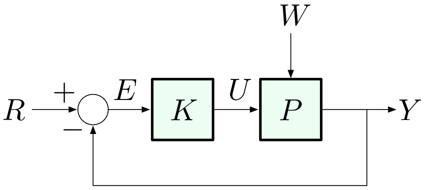
Reading: FPE, Section 4.1; lab manual

Two Basic Control Architectures

- Open-loop control

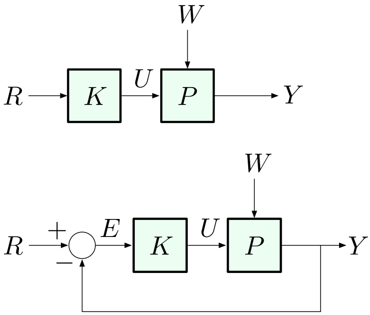


- Feedback (closed-loop) control



Here, W is a *disturbance*; K is *not necessarily* a static gain

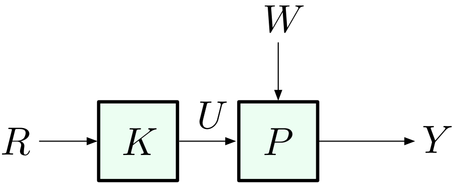
Basic Objectives of Control



- track a given reference
- reject disturbances
- meet performance specs

Intuitively, the difference between the open-loop and the closed-loop architectures is clear (think cruise control ...)

Open-Loop Control



- cheaper/easier to implement (no sensor required)
- does not destabilize the system

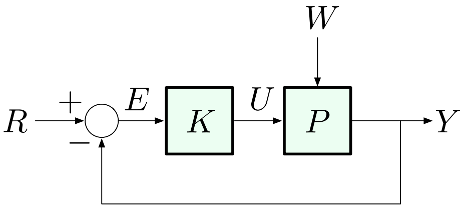
  e.g., if both K and P are stable (all poles in OLHP),

$$\frac{Y}{R} = KP$$

  is also stable:

$$\{\text{poles of } KP\} = \{\text{poles of } K\} \cup \{\text{poles of } P\}$$

Feedback Control

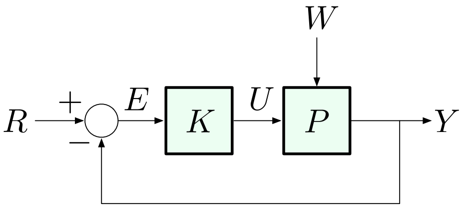


- more difficult/expensive to implement (requires a sensor and an information path from controller to actuator)
- may destabilize the system:

$$\frac{Y}{R} = \frac{KP}{1 + KP}$$

  has new poles, which may be unstable
- but: feedback control is the *only way* to stabilize an unstable plant (this was the Wright brothers' key insight)

Benefits of Feedback Control



Feedback control:

- reduces steady-state error to disturbances
- reduces steady-state sensitivity to model uncertainty (parameter variations)
- improves time response

Case Study: DC Motor

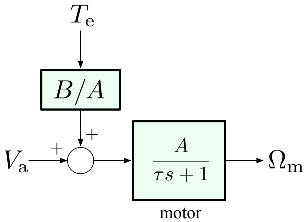
Inputs:	v_a – input voltage
	τ_e – load/disturbance torque
Outputs:	ω_m – angular speed of the motor

Transfer function:

$$\Omega_m = \frac{A}{\tau s + 1} V_a + \frac{B}{\tau s + 1} T_e$$

τ – time constant

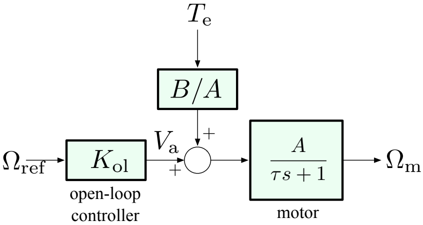
A, B – system gains



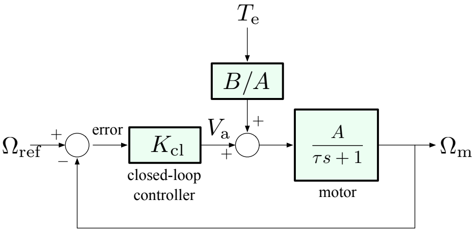
Objective: have Ω_m approach and track a given reference Ω_{ref} in spite of disturbance T_e.

Two Control Configurations

- Open-loop control



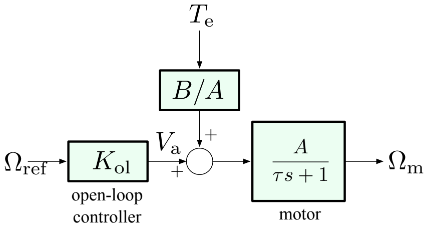
- Feedback (closed-loop) control



Disturbance Rejection

Goal: maintain $\omega_{\mathrm{m}} = \omega_{\mathrm{ref}}$ in steady state in the presence of *constant* disturbance.

Open-loop:

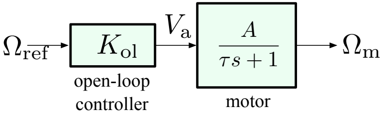


– the controller receives *no information* about the disturbance τ_{e} (the only input is ω_{ref}, no feedback signal from anywhere else)

– so, let's attempt the following: design for *no disturbance* (i.e., $\tau_{\mathrm{e}} = 0$), then see how the system works in general

Disturbance Rejection: Open-Loop Control

First assume zero disturbance:



Transfer function:

$$\frac{A}{\tau s+1} \text{ (stable pole at } s=-1/\tau\text{)}$$

We want DC gain $= 1$

$$\Omega_{\text{m}} = \frac{A}{\tau s+1} V_{\text{a}} = \frac{K_{\text{ol}} A}{\tau s+1} \Omega_{\text{ref}}$$

Let's just use constant gain: $K_{\text{ol}} = 1/A$

$$\omega_{\text{m}}(\infty) = \frac{1}{A} \cdot A \cdot \omega_{\text{ref}} = \omega_{\text{ref}} \qquad (\text{for } T_{\text{e}} = 0)$$

What happens in the presence of nonzero T_{e}?

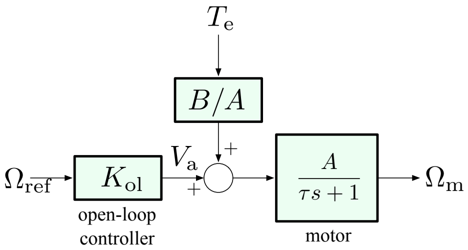
$$\Omega_{\text{m}} = \underbrace{\frac{A}{\tau s+1} \frac{1}{A}}_{\text{DC gain}=1} \Omega_{\text{ref}} + \underbrace{\frac{B}{\tau s+1}}_{\text{DC gain}=B} T_{\text{e}}$$

$$\Longrightarrow \omega_{\text{m}}(\infty) = \underbrace{\omega_{\text{ref}}}_{\text{step input}} + B \underbrace{\tau_{\text{e}}}_{\text{step input}}$$

Disturbance Rejection: Open-Loop Control

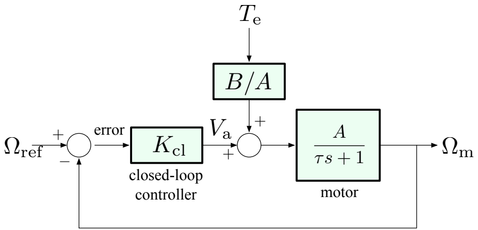
Steady-state motor speed for constant reference and constant disturbance:

$$\omega_{\mathrm{m}}(\infty) = \omega_{\mathrm{ref}} + B\tau_{\mathrm{e}}$$



Conclusion: in the absence of disturbances, reference tracking is good, but disturbance rejection is pretty poor. Steady-state error is determined by B, and we have no control over it (and, in fact, cannot change this through any choice of controller K_{ol}, no matter how clever)

Disturbance Rejection: Feedback Control



$$V_a = K_{cl}E = K_{cl}\left(\Omega_{ref} - \Omega_m\right)$$

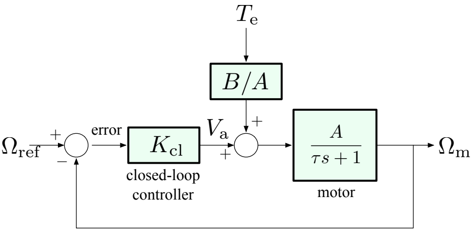
$$\Omega_m = \frac{A}{\tau s + 1}K_{cl}\left(\Omega_{ref} - \Omega_m\right) + \frac{B}{\tau s + 1}T_e$$

Solve for Ω_m: $\quad (\tau s + 1)\Omega_m = AK_{cl}\left(\Omega_{ref} - \Omega_m\right) + BT_e$

$$(\tau s + 1 + AK_{cl})\Omega_m = AK_{cl}\Omega_{ref} + BT_e$$

$$\Omega_m = \frac{AK_{cl}}{\tau s + 1 + AK_{cl}}\Omega_{ref} + \frac{B}{\tau s + 1 + AK_{cl}}T_e$$

Disturbance Rejection: Feedback Control



$$\Omega_m = \underbrace{\frac{AK_{cl}}{\tau s + 1 + AK_{cl}}}_{\text{DC gain}=\frac{AK_{cl}}{1+AK_{cl}}} \Omega_{ref} + \underbrace{\frac{B}{\tau s + 1 + AK_{cl}}}_{\text{DC gain}=\frac{B}{1+AK_{cl}}} T_e$$

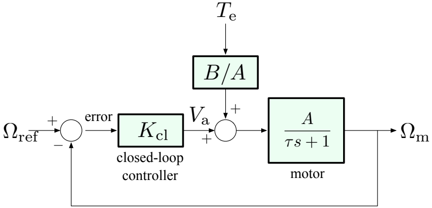
(provided all transfer functions are strictly stable)

Assuming that the reference ω_{ref} and the disturbance τ_e are constant, we apply FVT:

$$\omega_m(\infty) = \frac{AK_{cl}}{1 + AK_{cl}} \omega_{ref} + \frac{B}{1 + AK_{cl}} \tau_e$$

Disturbance Rejection: Feedback Control

Steady-state speed for constant reference and disturbance:



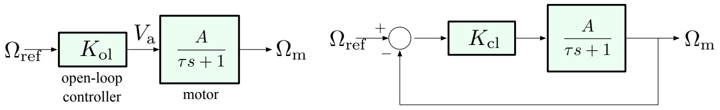
$$\omega_m(\infty) = \frac{AK_{cl}}{1+AK_{cl}}\omega_{ref} + \frac{B}{1+AK_{cl}}\tau_e$$

Conclusions:

- $\dfrac{AK_{cl}}{1+AK_{cl}} \neq 1$, but can be brought arbitrarily close to 1 when $K_{cl} \to \infty$. Thus, steady-state tracking is good with high gain, but never quite as good as in open-loop case.
- $\dfrac{B}{1+AK_{cl}}$ is small (arbitrarily close to 0) for large K_{cl}. Thus, *much* better disturbance rejection than with open-loop control.

Sensitivity to Parameter Variations

Consider again our DC motor model, with no disturbance:



Bode's sensitivity concept: In the "nominal" situation, we have the motor with DC gain $= A$, and the overall transfer function, either open- or closed-loop, has some other DC gain (call it T).

Now suppose that, due to modeling error, changes in operating conditions, etc., the motor gain changes:

$$A \longrightarrow A + \underbrace{\delta A}_{\substack{\text{small}\\ \text{perturbation}}}$$

This will cause a perturbation in the overall DC gain:

$$T \longrightarrow T + \delta T \qquad \text{(from calculus, to 1st order, } \delta T \approx \frac{\mathrm{d}T}{\mathrm{d}A}\delta A\text{)}$$

Sensitivity to Parameter Variations

$$A \longrightarrow A + \delta A \qquad \text{(small perturbation in system gain)}$$
$$T \longrightarrow T + \delta T \qquad \text{(resultant perturbation in overall DC gain)}$$

Hendrik Wade Bode
(1905–1982)

Bode's sensitivity:

$$\mathcal{S} \triangleq \frac{\delta T/T}{\delta A/A}$$

$$\mathcal{S} = \text{relative error}$$
$$= \frac{\text{normalized (percentage) error in } T}{\text{normalized (percentage) error in } A}$$

Sensitivity to Parameter Variations

Let's compute $\mathcal{S}$ for our DC motor control example, both open- and closed-loop.

Open-loop:

- nominal case $\qquad T_{\text{ol}} = K_{\text{ol}} A = \frac{1}{A} A = 1$
- perturbed case

$$
\begin{aligned}
A &\longrightarrow A + \delta A \\
T_{\text{ol}} &\longrightarrow K_{\text{ol}}(A + \delta A) = \underbrace{\frac{1}{A}}_{\substack{\text{design}\\ \text{choice}}} (A + \delta A) = \underbrace{1}_{T_{\text{ol}}} + \underbrace{\frac{\delta A}{A}}_{\delta T_{\text{ol}}}
\end{aligned}
$$

Sensitivity: $\quad \mathcal{S}_{\text{ol}} = \dfrac{\delta T_{\text{ol}}/T_{\text{ol}}}{\delta A_{\text{ol}}/A_{\text{ol}}} = \dfrac{\delta A/A}{\delta A/A} = 1$

For example, a 5% error in A will cause a 5% error in T_{ol}.

Sensitivity to Parameter Variations

Closed-loop:

- nominal case $\quad T_{\mathrm{cl}} = \dfrac{AK_{\mathrm{cl}}}{1 + AK_{\mathrm{cl}}}$
- perturbed case

$$A \longrightarrow A + \delta A \qquad T_{\mathrm{cl}} \longrightarrow T_{\mathrm{cl}} + \underbrace{\delta T_{\mathrm{cl}}}_{\substack{\text{how to} \\ \text{compute this?}}}$$

Taylor expansion:

$$T(A + \delta A) = T(A) + \frac{\mathrm{d}T}{\mathrm{d}A}(A)\delta A + \text{higher-order terms}$$

In our case:

$$\frac{\mathrm{d}T_{\mathrm{cl}}}{\mathrm{d}A} = \frac{K_{\mathrm{cl}}}{1 + AK_{\mathrm{cl}}} - \frac{AK_{\mathrm{cl}}^2}{(1 + AK_{\mathrm{cl}})^2} = \frac{K_{\mathrm{cl}}}{(1 + AK_{\mathrm{cl}})^2}$$

$$\delta T_{\mathrm{cl}} = \frac{K_{\mathrm{cl}}}{(1 + AK_{\mathrm{cl}})^2}\delta A$$

Sensitivity to Parameter Variations

From before:

$$\delta T_{\mathrm{cl}} = \frac{K_{\mathrm{cl}}}{(1 + AK_{\mathrm{cl}})^2}\delta A$$

$$T_{\mathrm{cl}} = \frac{AK_{\mathrm{cl}}}{1 + AK_{\mathrm{cl}}}$$

Therefore

$$\delta T_{\mathrm{cl}}/T_{\mathrm{cl}} = \frac{\frac{K_{\mathrm{cl}}}{(1+AK_{\mathrm{cl}})^2}\delta A}{\frac{AK_{\mathrm{cl}}}{1+AK_{\mathrm{cl}}}} = \frac{1}{1 + AK_{\mathrm{cl}}}\delta A/A$$

Sensitivity: $\mathcal{S}_{\mathrm{cl}} = \dfrac{\delta T_{\mathrm{cl}}/T_{\mathrm{cl}}}{\delta A/A} = \dfrac{1}{1 + AK_{\mathrm{cl}}}$ $\quad (\ll 1$ for large $K_{\mathrm{cl}})$

With high-gain feedback, we get smaller relative error due to parameter variations in the plant model.

Time Response

We still assume no disturbance: $\tau_\mathrm{e} = 0$.

So far, we have focused on DC gain only (steady-state response). What about *transient response*?

Open-loop

$$\Omega_\mathrm{m} = \frac{AK_\mathrm{cl}}{\tau s + 1}\Omega_\mathrm{ref}$$

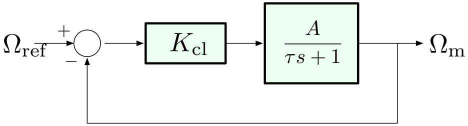
Pole at $s = -\frac{1}{\tau}$ $\Longrightarrow$ transient response is $e^{-t/\tau}$

Here, τ is the *time constant*: the time it takes the system response to decay to $1/e$ of its starting value.

In the open-loop case, larger time constant means faster convergence to steady state. This is not affected by the choice of K_cl in any way!

Time Response

Closed-loop



$$\Omega_{\text{m}} = \frac{AK_{\text{cl}}}{\tau s + 1 + AK_{\text{cl}}}\Omega_{\text{ref}}$$

Closed-loop pole at $s = -\frac{1}{\tau}\left(1 + AK_{\text{cl}}\right)$
(the only way to move poles around is *via feedback*)

Now the transient response is $e^{-\frac{1+AK_{\text{cl}}}{\tau}t}$, with

$$\text{time constant} = \frac{\tau}{1 + AK_{\text{cl}}}$$

— for large K_{cl}, we have a much smaller time constant, i.e., *faster convergence* to steady-state.

Summary

Feedback control:

- reduces steady-state error to disturbances
- reduces steady-state sensitivity to model uncertainty (parameter variations)
- improves time response

Word of caution: high-gain feedback only works well for 1st-order systems; for higher-order systems, it will typically cause underdamping and instability.

Thus, we need a more sophisticated design than just static gain. We will take this up in the next lecture with *Proportional-Integral-Derivative* (PID) control.