Plan of the Lecture

- Review: transient and steady-state response; DC gain and the FVT
- Today's topic: system-modeling diagrams; prototype 2nd-order system

Goal: develop a methodology for representing and analyzing systems by means of block diagrams; start analyzing a prototype 2nd-order system.

Reading: FPE, Sections 3.1–3.2; lab manual

System Modeling Diagrams

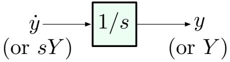
$$\text{large system} \underset{\text{compose}}{\overset{\text{decompose}}{\rightleftharpoons}} \text{smaller blocks (subsystems)}$$

— this is the core of systems theory

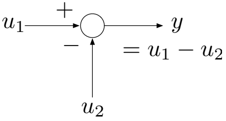
We will take smaller blocks from some given *library* and play with them to create/build more complicated systems.

All-Integrator Diagrams

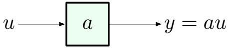
Our library will consist of three building blocks:



integrator



summing junction



constant gain

Two warnings:

- We can (and will) work either with u, y (time domain) or with U, Y (s-domain) — will often go back and forth
- When working with block diagrams, we typically ignore initial conditions.

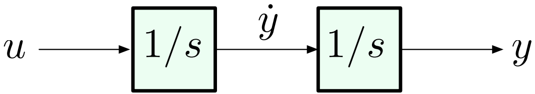
This is the *lowest level* we will go to in lectures; in the labs, you will implement these blocks using op amps.

Example 1

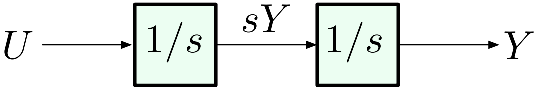
Build an all-integrator diagram for

$$\ddot{y} = u \qquad \Longleftrightarrow \qquad s^2 Y = U$$

This is obvious:



or



Example 2

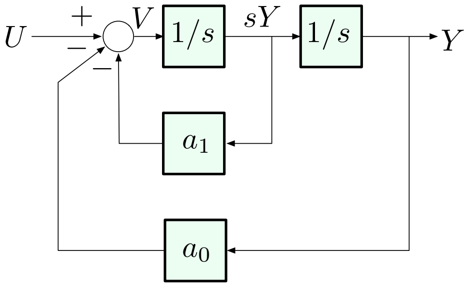
(building on Example 1)

$$\ddot{y} + a_1\dot{y} + a_0 y = u \qquad \Longleftrightarrow \qquad s^2 Y + a_1 sY + a_0 Y = U$$

$$\text{or} \qquad Y(s) = \frac{U(s)}{s^2 + a_1 s + a_0}$$

Always solve for the highest derivative:

$$\ddot{y} = \underbrace{-a_1\dot{y} - a_0 y + u}_{=v}$$

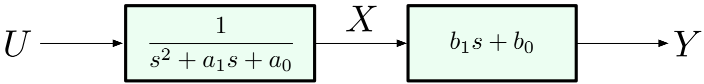


Example 3

Build an all-integrator diagram for a system with transfer function

$$H(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}$$

Step 1: decompose $H(s) = \frac{1}{s^2 + a_1 s + a_0} \cdot (b_1 s + b_0)$

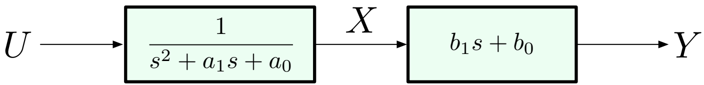


— here, X is an auxiliary (or intermediate) signal

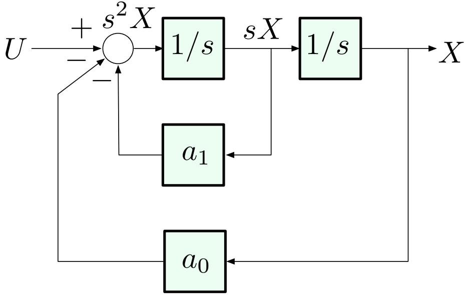
Note: $b_0 + b_1 s$ involves *differentiation*, which we cannot implement using an all-integrator diagram. But we will see that we don't need to do it directly.

Example 3, continued

Step 1: decompose $H(s) = \frac{1}{s^2 + a_1 s + a_0} \cdot (b_1 s + b_0)$



Step 2: The transformation $U \to X$ is from Example 2:

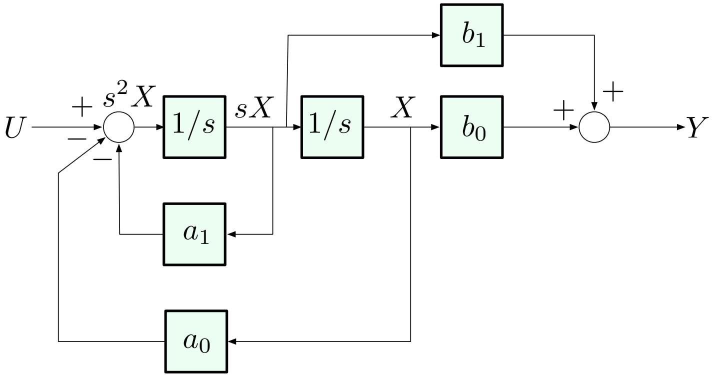


Example 3, continued

Step 3: now we notice that

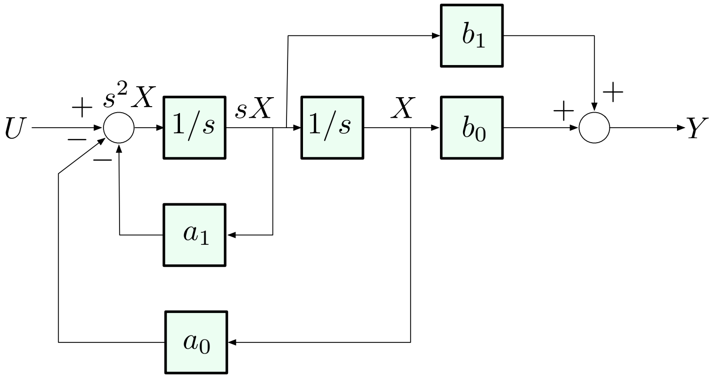
$$Y(s) = b_1 sX(s) + b_0 X(s),$$

and both X and sX are available signals in our diagram. So:



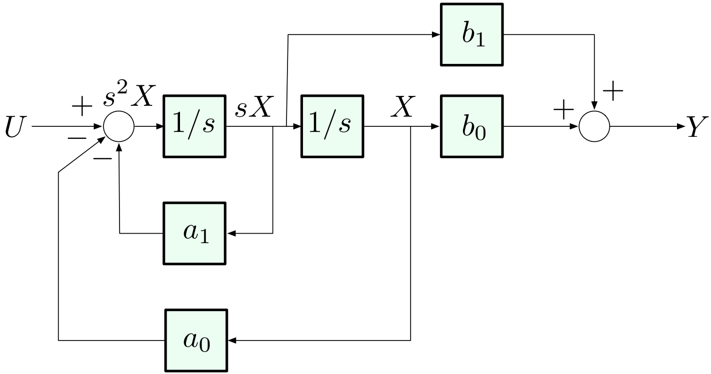
Example 3, continued

All-integrator diagram for $H(s) = \dfrac{b_1 s + b_0}{s^2 + a_1 s + a_0}$



Can we write down a state-space model corresponding to this diagram?

Example 3, continued



State-space model:

$$s^2X = U - a_1 sX - a_0 X \qquad Y = b_1 sX + b_0 X$$

$$\ddot{x} = -a_1\dot{x} - a_0 x + u \qquad y = b_1\dot{x} + b_0 x$$

Example 3, continued

State-space model:

$$\ddot{x} = -a_1\dot{x} - a_0 x + u \qquad y = b_1\dot{x} + b_0 x$$

$$x_1 = x,\ x_2 = \dot{x}$$

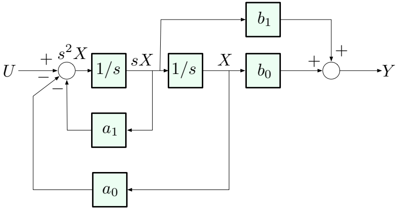
$$\begin{pmatrix}\dot{x}_1 \\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 1 \\ -a_0 & -a_1\end{pmatrix}\begin{pmatrix}x_1 \\ x_2\end{pmatrix} + \begin{pmatrix}0 \\ 1\end{pmatrix} u \qquad y = \begin{pmatrix}b_0 & b_1\end{pmatrix}\begin{pmatrix}x_1 \\ x_2\end{pmatrix}$$

This is called *controller canonical form*.

- Easily generalizes to dimension > 1
- The reason behind the name will be made clear later in the semester

Example 3, wrap-up

All-integrator diagram for $H(s) = \dfrac{b_1 s + b_0}{s^2 + a_1 s + a_0}$



State-space model:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \qquad y = \begin{pmatrix} b_0 & b_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Important: for a given $H(s)$, the diagram is *not unique*. But, once we build a diagram, the state-space equations are unique (up to coordinate transformations).

Basic System Interconnections

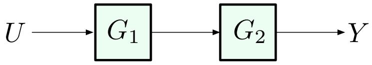
Now we will take this a level higher — we will talk about building complex systems from smaller blocks, without worrying about how those blocks look on the inside (they could themselves be all-integrator diagrams, etc.)

Block diagrams are an *abstraction* (they hide unnecessary "low-level" detail ...)

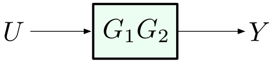
Block diagrams describe the *flow of information*

Basic System Interconnections: Series & Parallel

Series connection



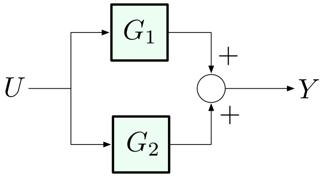
(G is common notation for t.f.'s)



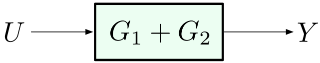
$$\frac{Y}{U} = G_1G_2$$

(for SISO systems, the order of G_1 and G_2 does not matter)

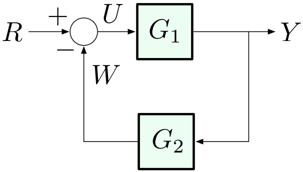
Parallel connection



$$\frac{Y}{U} = G_1 + G_2$$



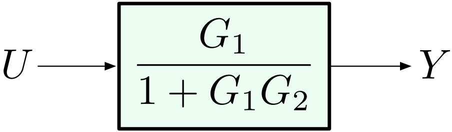
Basic System Interconnections: Negative Feedback



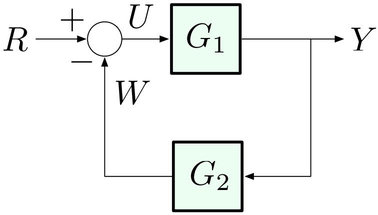
Find the transfer function from R (reference) to Y

$$\implies Y = \frac{G_1}{1 + G_1 G_2} R$$

$$
\begin{aligned}
U &= R - W \\
Y &= G_1 U \\
&= G_1 (R - W) \\
&= G_1 R - G_1 G_2 Y
\end{aligned}
$$



Basic System Interconnections: Negative Feedback



$$\implies Y = \frac{G_1}{1 + G_1 G_2} R$$

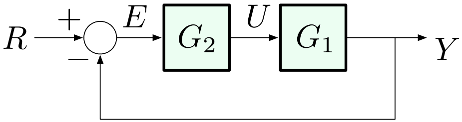
The gain of a negative feedback loop:

$$\frac{\text{forward gain}}{1 + \text{loop gain}}$$

This is an important relationship, easy to derive — no need to memorize it.

Unity Feedback

Other feedback configurations are also possible:

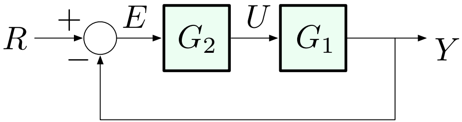


This is called *unity feedback* — no component on the feedback path.

Common structure (saw this in Lecture 1):

- R = reference
- U = control input
- Y = output
- E = error
- G_1 = plant (also denoted by P)
- G_2 = controller or compensator (also denoted by C or K)

Unity Feedback



Let's practice with deriving transfer functions: $\dfrac{\text{forward gain}}{1 + \text{loop gain}}$

- Reference R to output Y:

$$\frac{Y}{R} = \frac{G_1 G_2}{1 + G_1 G_2}$$

- Reference R to control input U:

$$\frac{U}{R} = \frac{G_2}{1 + G_1 G_2}$$

- Error E to output Y:

$$\frac{Y}{E} = G_1 G_2 \qquad \text{(no feedback path)}$$

Block Diagram Reduction

Given a complicated diagram involving series, parallel, and feedback interconnections, we often want to write down an overall transfer function from one of the variables to another.

This requires lots of practice: read FPE, Section 3.2 for examples.

General strategy:

- Name all the variables in the diagram
- Write down as many relationships between these variables as you can
- Learn to recognize series, parallel, and feedback interconnections
- Replace them by their equivalents
- Repeat

Prototype 2nd-Order System

So far, we have only seen transfer functions that have either real poles or purely imaginary poles:

$$\frac{1}{s+a}, \qquad \frac{1}{(s+a)(s+b)}, \qquad \frac{1}{s^2+\omega^2}$$

We also need to consider the case of *complex poles*, i.e., ones that have $\mathrm{Re}(s) \neq 0$ and $\mathrm{Im}(s) \neq 0$.

For now, we will only look at *second-order systems*, but this will be sufficient to develop some nontrivial intuition (dominant poles).

Plus, you will need this for Lab 1.

Prototype 2nd-Order System

Consider the following transfer function:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Comments:

- $\zeta > 0, \omega_n > 0$ are arbitrary parameters
- the denominator is a general 2nd-degree monic polynomial, just written in a weird way
- $H(s)$ is normalized to have DC gain $= 1$ (provided DC gain exists)

Prototype 2nd-Order System

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

By the quadratic formula, the poles are:

$$\begin{aligned} s &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \\ &= -\omega_n\left(\zeta \pm \sqrt{\zeta^2 - 1}\right) \end{aligned}$$

The nature of the poles changes depending on ζ:

- $\zeta > 1$ both poles are real and negative
- $\zeta = 1$ one negative pole
- $\zeta < 1$ two complex poles with negative real parts

$$s = -\sigma \pm j\omega_d$$

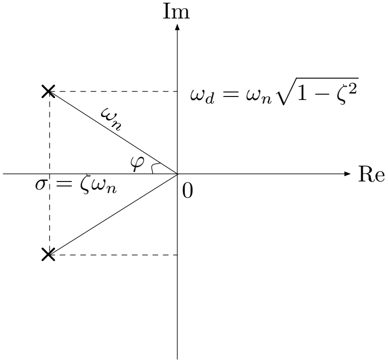
$$\text{where} \qquad \sigma = \zeta\omega_n,\ \omega_d = \omega_n\sqrt{1 - \zeta^2}$$

Prototype 2nd-Order System

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \qquad \zeta < 1$$

The poles are

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\sigma \pm j\omega_d$$



Note that

$$\begin{aligned}\sigma^2 + \omega_d^2 &= \zeta^2\omega_n^2 + \omega_n^2 - \zeta^2\omega_n^2 \\ &= \omega_n^2 \\ \cos\varphi &= \frac{\zeta\omega_n}{\omega_n} = \zeta\end{aligned}$$

2nd-Order Response

Let's compute the system's impulse and step response:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

- Impulse response:

$$h(t) = \mathscr{L}^{-1}\{H(s)\} = \mathscr{L}^{-1}\left\{\frac{(\omega_n^2/\omega_d)\omega_d}{(s+\sigma)^2 + \omega_d^2}\right\}$$

$$= \frac{\omega_n^2}{\omega_d} e^{-\sigma t} \sin(\omega_d t) \qquad \text{(table, \# 20)}$$

- Step response:

$$\mathscr{L}^{-1}\left\{\frac{H(s)}{s}\right\} = \mathscr{L}^{-1}\left\{\frac{\sigma^2 + \omega_d^2}{s[(s+\sigma)^2 + \omega_d^2]}\right\}$$

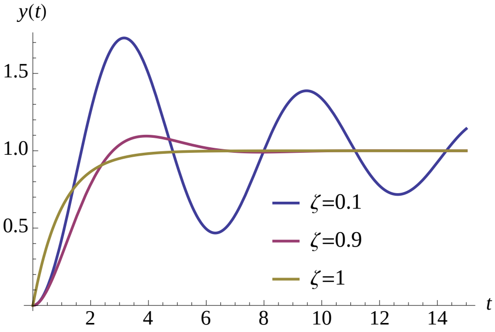
$$= 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right) \qquad \text{(table, \#21)}$$

2nd-Order Step Response

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

$$u(t) = 1(t) \qquad \longrightarrow \qquad y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

where $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$ (damped frequency)



The parameter ζ is called the *damping ratio*

- $\zeta > 1$: system is overdamped
- $\zeta < 1$: system is underdamped
- $\zeta = 0$: no damping ($\omega_d = \omega_n$)