

## Plan of the Lecture

- ▶ **Review:** transient response specs (rise time, overshoot, settling time)
- ▶ **Today's topic:** effect of zeros and extra poles; Routh–Hurwitz stability criterion

*Goal:* understand the effect of zeros and high-order poles on the shape of transient response; discuss relation with stability; formulate and learn how to apply the Routh–Hurwitz stability criterion.

*Reading:* FPE, Sections 3.5–3.6

## Effect of Zeros on the Transient Response

**Reminder:** for  $H(s) = \frac{q(s)}{p(s)}$ , zeros are the roots of  $q(s) = 0$

**Example:** start with  $H_1(s) = \frac{1}{s^2 + 2\zeta s + 1}$  ( $\omega_n = 1$ )

Let's add a zero at  $s = -a$ ,  $a > 0$  – LHP zero

To keep DC gain = 1, let's take the numerator to be  $\frac{s}{a} + 1$ :

$$\begin{aligned} H_2(s) &= \frac{\frac{s}{a} + 1}{s^2 + 2\zeta s + 1} \\ &= \underbrace{\frac{1}{s^2 + 2\zeta s + 1}}_{\text{this is } H_1(s)} + \frac{1}{a} \cdot \underbrace{\frac{s}{s^2 + 2\zeta s + 1}}_{\text{call this } H_d(s)} \\ &= H_1(s) + \frac{1}{a} H_d(s), \quad H_d(s) = sH_1(s) \end{aligned}$$

## Effect of a LHP Zero

$$H_1(s) = \frac{1}{s^2 + 2\zeta s + 1} \xrightarrow{\text{add zero at } s = -a} H_2(s) = H_1(s) + \frac{1}{a} \cdot sH_1(s)$$

Step response:

$$\begin{aligned} Y_1(s) &= \frac{H_1(s)}{s} \\ Y_2(s) &= \frac{H_2(s)}{s} \\ &= \frac{H_1(s)}{s} + \frac{1}{a} \frac{sH_1(s)}{s} \\ &= Y_1(s) + \frac{1}{a} sY_1(s) \end{aligned}$$

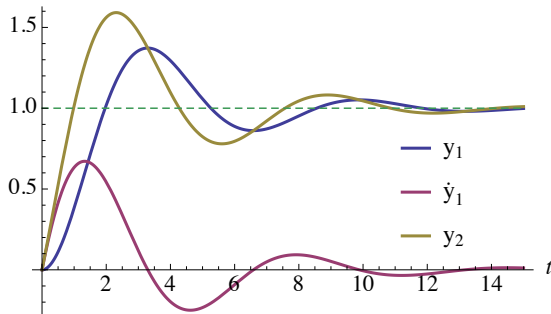
$$y_2(t) = \mathcal{L}^{-1}\{Y_2(s)\} = \mathcal{L}^{-1}\left\{Y_1(s) + \frac{1}{a} \cdot sY_1(s)\right\} = y_1(t) + \frac{1}{a} \dot{y}_1(t)$$

(assuming zero initial conditions)

## Effect of a LHP Zero

Step response (zero at  $s = -a$ )

$$y_2(t) = y_1(t) + \frac{1}{a} \dot{y}_1(t) \quad \text{where } y_1(t) = \text{original step response}$$

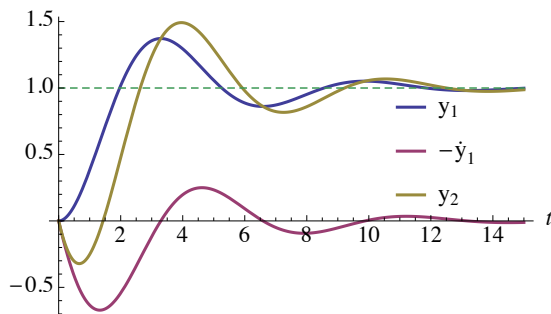


Effects of a LHP zero:

- ▶ increased overshoot (major effect)
- ▶ little influence on settling time
- ▶ what happens as  $a \rightarrow \infty$ ? — effects become less significant

## What About a RHP Zero?

$$H_1(s) = \frac{1}{s^2 + 2\zeta s + 1} \xrightarrow{\text{add zero at } s = a} H_2(s) = H_1(s) - \frac{1}{a} \cdot sH_1(s)$$
$$y_2(t) = y_1(t) - \frac{1}{a} \cdot \dot{y}_1(t)$$

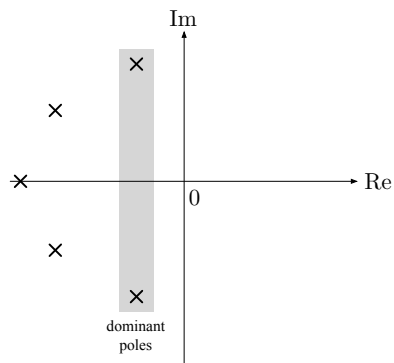


Effects of a RHP zero:

- ▶ slows down (delays) the response
- ▶ creates *undershoot* (at least, when  $a$  is small enough)

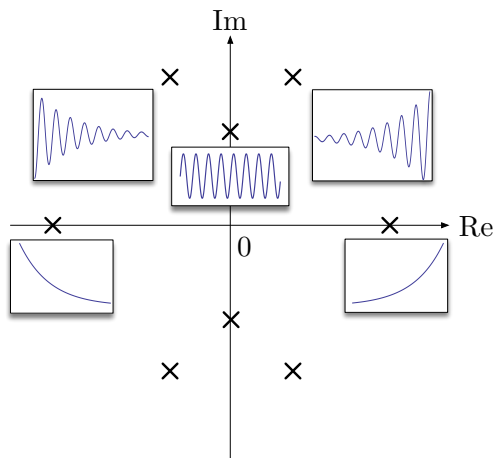
## Effect of Extra Poles

A general  $n$ th-order system has  $n$  poles



- ▶ extra LHP poles are not significant if their real parts are at least  $5\times$  the real parts of dominant LHP poles
- ▶ e.g., if dominant poles have  $\text{Re}(s) = -2$  and we have extra poles with  $\text{Re}(s) = -10$ , their time-domain contributions will be  $e^{-2t}$  and  $e^{-10t} \ll e^{-2t}$
- ▶  $5\times$  is just a convention, but we can really see the effect of extra poles that are closer (cf. Lab 2)

## Effect of Pole Locations



- ▶ poles in open LHP ( $\text{Re}(s) < 0$ ) — stable response
- ▶ poles in open RHP ( $\text{Re}(s) > 0$ ) — unstable response
- ▶ poles on the imaginary axis ( $\text{Re}(s) = 0$ ) — tricky case

## Marginal Case: Poles on the Imaginary Axis

Let's consider the case of a pole at the origin:  $H(s) = \frac{1}{s}$

Is this a stable system?

- ▶ impulse response:  $Y(s) = \frac{1}{s} \implies y(t) = 1(t)$  (OK)
- ▶ step response:  $Y(s) = \frac{1}{s^2} \implies y(t) = t, t \geq 0$  — *unit ramp!!*

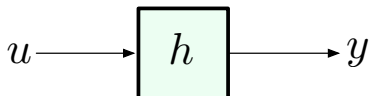
What about purely imaginary poles?  $H(s) = \frac{\omega^2}{s^2 + \omega^2}$

- ▶ impulse response:  $Y(s) = \frac{\omega^2}{s^2 + \omega^2} \implies y(t) = \omega \sin(\omega t)$
- ▶ step response:  $Y(s) = \frac{\omega^2}{s(s^2 + \omega^2)} \implies y(t) = 1 - \cos(\omega t)$

Systems with poles on the imaginary axis are *not stable*.



## What Is Stability?



One reasonable definition is as follows:

A linear time-invariant system is *Bounded-Input, Bounded-Output (BIBO) stable* provided either one of the following three equivalent conditions is satisfied:

1. If every bounded input  $u(t)$  results in a bounded output  $y(t)$ , regardless of initial conditions.
2. If the impulse response  $h(t)$  is absolutely integrable:

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty.$$

3. If all poles of the transfer function  $H(s)$  are *strictly stable* (lie in open LHP).

## Checking for Stability?

Consider a general transfer function:

$$H(s) = \frac{q(s)}{p(s)}$$

where  $q$  and  $p$  are polynomials, and  $\deg(q) \leq \deg(p)$ .

We need tools for checking stability: whether or not all roots of  $p(s) = 0$  lie in OLHP.

For simple polynomials, can just factor them “by inspection” and find roots.

Now, this is hard to do for high-degree polynomials — it’s computationally intensive, especially symbolically.

But: often we *don’t need to know* precise pole locations, just need to know that they are **strictly stable**.

## Checking for Stability

**Problem:** given an  $n$ th-degree polynomial

$$p(s) = s^n + a_1s^{n-1} + a_2s^{n-2} + \dots + a_{n-1}s + a_n$$

with real coefficients, check that the roots of the equation  $p(s) = 0$  are strictly stable (i.e., have negative real parts).

**Terminology:** we often say that the polynomial  $p$  is (strictly) stable if all of its roots are.

## A Necessary Condition for Stability

Terminology: we say that  $A$  is a **necessary condition** for  $B$  if

$$A \text{ is false} \implies B \text{ is false}$$

*Important!!* Even if  $A$  is true,  $B$  may still be false.

**Necessary condition for stability:** a polynomial  $p$  is strictly stable only if all of its coefficients are strictly positive.

**Proof:** suppose that  $p$  has roots at  $r_1, r_2, \dots, r_n$  with  $\operatorname{Re}(r_i) < 0$  for all  $i$ . Then

$$p(s) = (s - r_1)(s - r_2) \dots (s - r_n)$$

— multiply this out and check that all coefficients are positive.

# Routh–Hurwitz Criterion

## Necessary & Sufficient Condition for Stability

Terminology: we say that  $A$  is a **sufficient condition** for  $B$  if

$$A \text{ is true} \implies B \text{ is true}$$

Thus,  $A$  is a **necessary and sufficient condition** for  $B$  if

$$A \text{ is true} \iff B \text{ is true}$$

— we also say that  $A$  is true **if and only if** (iff)  $B$  is true.

We will now introduce a necessary and sufficient condition for stability: the *Routh–Hurwitz Criterion*.

## Routh–Hurwitz Criterion: A Bit of History

J.C. Maxwell, “On governors,” Proc. Royal Society, no. 100, 1868

... [Stability of the governor] is mathematically equivalent to the condition that all the possible roots, and all the possible parts of the impossible roots, of a certain equation shall be negative. ...

I have not been able completely to determine these conditions for equations of a higher degree than the third; but I hope that the subject will obtain the attention of mathematicians.



In 1877, Maxwell was one of the judges for the Adams Prize, a biennial competition for best essay on a scientific topic. The topic that year was [stability of motion](#). The prize went to [Edward John Routh](#), who solved the problem posed by Maxwell in 1868.

In 1893, [Adolf Hurwitz](#) solved the same problem, using a different method, independently of Routh.



Edward John Routh, 1831–1907



Adolf Hurwitz, 1859–1919

# Routh's Test

**Problem:** check whether the polynomial

$$p(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} s + a_n$$

is strictly stable.

We begin by forming the **Routh array** using the coefficients of  $p$ :

$$\begin{array}{l} s^n : \quad 1 \quad a_2 \quad a_4 \quad a_6 \quad \dots \\ s^{n-1} : \quad a_1 \quad a_3 \quad a_5 \quad a_7 \quad \dots \end{array} \quad \begin{array}{l} \text{(if necessary, add zeros in the} \\ \text{second row to match lengths)} \end{array}$$

Note that the very first entry is always 1, and also note the order in which the coefficients are filled in.



## Routh's Test

$$\begin{array}{l} s^n : \quad 1 \quad a_2 \quad a_4 \quad a_6 \quad \dots \\ s^{n-1} : \quad a_1 \quad a_3 \quad a_5 \quad a_7 \quad \dots \\ s^{n-2} : \quad b_1 \quad b_2 \quad b_3 \quad \dots \end{array}$$

Next, we form the third row marked by  $s^{n-2}$ :

$$s^{n-2} : \quad b_1 \quad b_2 \quad b_3 \quad \dots$$

$$\text{where } b_1 = -\frac{1}{a_1} \det \begin{pmatrix} 1 & a_2 \\ a_1 & a_3 \end{pmatrix} = -\frac{1}{a_1} (a_3 - a_1 a_2)$$

$$b_2 = -\frac{1}{a_1} \det \begin{pmatrix} 1 & a_4 \\ a_1 & a_5 \end{pmatrix} = -\frac{1}{a_1} (a_5 - a_1 a_4)$$

$$b_3 = -\frac{1}{a_1} \det \begin{pmatrix} 1 & a_6 \\ a_1 & a_7 \end{pmatrix} = -\frac{1}{a_1} (a_7 - a_1 a_6) \quad \text{and so on ...}$$

**Note:** the new row is 1 element shorter than the one above it

## Routh's Test, continued

$$\begin{array}{l} s^n : \quad 1 \quad a_2 \quad a_4 \quad a_6 \quad \dots \\ s^{n-1} : \quad a_1 \quad a_3 \quad a_5 \quad a_7 \quad \dots \\ s^{n-2} : \quad b_1 \quad b_2 \quad b_3 \quad \dots \\ s^{n-3} : \quad c_1 \quad c_2 \quad \dots \end{array}$$

Next, we form the fourth row marked by  $s^{n-3}$ :

$$s^{n-3} : \quad c_1 \quad c_2 \quad \dots$$

where

$$c_1 = -\frac{1}{b_1} \det \begin{pmatrix} a_1 & a_3 \\ b_1 & b_2 \end{pmatrix} = -\frac{1}{b_1} (a_1 b_2 - a_3 b_1)$$
$$c_2 = -\frac{1}{b_1} \det \begin{pmatrix} a_1 & a_5 \\ b_1 & b_3 \end{pmatrix} = -\frac{1}{b_1} (a_1 b_3 - a_5 b_1)$$

and so on ...

## Routh's Test, continued

Eventually, we complete the array like this:

$$\begin{array}{l} s^n : \quad 1 \quad a_2 \quad a_4 \quad a_6 \quad \dots \\ s^{n-1} : \quad a_1 \quad a_3 \quad a_5 \quad a_7 \quad \dots \\ s^{n-2} : \quad b_1 \quad b_2 \quad b_3 \quad \dots \\ s^{n-3} : \quad c_1 \quad c_2 \quad \dots \\ \vdots \\ s^1 : \quad * \quad * \\ s^0 : \quad * \end{array} \quad \begin{array}{l} \\ \\ \text{(as long as we don't get stuck with} \\ \\ \text{division by zero: more on this later)} \end{array}$$

After the process terminates, we will have  $n + 1$  entries in the first column.

# The Routh–Hurwitz Criterion

Consider degree- $n$  polynomial

$$p(s) = s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n$$

and form the Routh array:

$$\begin{array}{l} s^n : \quad 1 \quad a_2 \quad a_4 \quad a_6 \quad \dots \\ s^{n-1} : \quad a_1 \quad a_3 \quad a_5 \quad a_7 \quad \dots \\ s^{n-2} : \quad b_1 \quad b_2 \quad b_3 \quad \dots \\ s^{n-3} : \quad c_1 \quad c_2 \quad \dots \\ \vdots \\ s^1 : \quad * \quad * \\ s^0 : \quad * \end{array}$$

**The Routh–Hurwitz criterion:** Assume that the necessary condition for stability holds, i.e.,  $a_1, \dots, a_n > 0$ . Then:

- ▶  $p$  is stable if and only if all entries in the first column are positive;
- ▶ otherwise,  $\#(\text{RHP poles}) = \#(\text{sign changes in 1st column})$

## Example

Check stability of

$$p(s) = s^4 + 4s^3 + s^2 + 2s + 3$$

All coefficients strictly positive: necessary condition checks out.

$$\begin{array}{l} s^4 : \quad 1 \quad 1 \quad 3 \\ s^3 : \quad 4 \quad 2 \quad 0 \\ s^2 : \quad 1/2 \quad 3 \\ s^1 : \quad -22 \quad 0 \\ s^0 : \quad 3 \end{array}$$

**Answer:**  $p$  is unstable — it has 2 RHP poles (2 sign changes in 1st column)

## Low-Order Cases ( $n = 2, 3$ )

$$n = 2 \quad p(s) = s^2 + a_1s + a_2$$

$$s^2 \quad : 1 \quad a_2$$

$$s^1 \quad : a_1 \quad 0$$

$$s^0 : \quad b_1$$

$$b_1 = -\frac{1}{a_1} \det \begin{pmatrix} 1 & a_2 \\ a_1 & 0 \end{pmatrix} = a_2$$

—  $p$  is stable iff  $a_1, a_2 > 0$  (necessary *and* sufficient).

$$n = 3 \quad p(s) = s^3 + a_1s^2 + a_2s + a_3$$

$$s^3 \quad : 1 \quad a_2$$

$$s^2 \quad : a_1 \quad a_3$$

$$s^1 : \quad b_1 \quad 0$$

$$s^0 : \quad c_1$$

$$b_1 = -\frac{1}{a_1} \det \begin{pmatrix} 1 & a_2 \\ a_1 & a_3 \end{pmatrix} = \frac{a_1a_2 - a_3}{a_1}$$

$$c_1 = -\frac{1}{b_1} \det \begin{pmatrix} a_1 & a_3 \\ b_1 & 0 \end{pmatrix} = a_3$$

—  $p$  is stable iff  $a_1, a_2, a_3 > 0$  (necc. cond.) and  $a_1a_2 > a_3$

# Stability Conditions for Low-Order Polynomials

## The upshot:

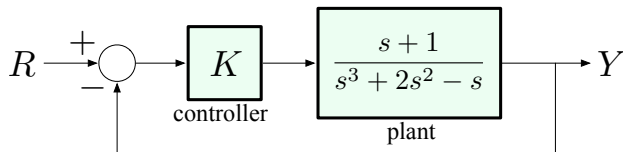
- ▶ A 2nd-degree polynomial  $p(s) = s^2 + a_1s + a_2$  is stable if and only if  $a_1 > 0$  and  $a_2 > 0$
  - ▶ A 3rd-degree polynomial  $p(s) = s^3 + a_1s^2 + a_2s + a_3$  is stable if and only if  $a_1, a_2, a_3 > 0$  and  $a_1a_2 > a_3$
- 
- ▶ These conditions were already obtained by Maxwell in 1868.
  - ▶ In both cases, the computations were *purely symbolic*: this can make a lot of difference in *design*, as opposed to *analysis*.

# Routh–Hurwitz as a Design Tool

## Parametric stability range

We can use the Routh test to determine *parameter ranges* for stability.

**Example:** consider the unity feedback configuration

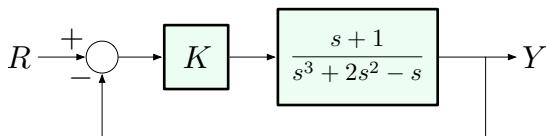


Note that the plant is *unstable* (the denominator has a negative coefficient and a zero coefficient).

**Problem:** determine the range of values the *scalar gain*  $K$  can take, for which the closed-loop system is stable.



## Example, continued



**Problem:** determine the range of values the **scalar gain**  $K$  can take, for which the closed-loop system is stable.

Let's write down the transfer function from  $R$  to  $Y$ :

$$\begin{aligned}\frac{Y}{R} &= \frac{\text{forward gain}}{1 + \text{loop gain}} \\ &= \frac{K \cdot \frac{s+1}{s^3+2s^2-s}}{1 + K \cdot \frac{s+1}{s^3+2s^2-s}} = \frac{K(s+1)}{s^3 + 2s^2 - s + K(s+1)} \\ &= \frac{Ks + K}{s^3 + 2s^2 + (K-1)s + K}\end{aligned}$$

Now we need to test stability of  $p(s) = s^3 + 2s^2 + (K-1)s + K$ .

## Example, continued

Test stability of

$$p(s) = s^3 + 2s^2 + (K - 1)s + K$$

using the Routh test.

Form the Routh array:

$$\begin{array}{l} s^3 : \quad 1 \quad K - 1 \\ s^2 : \quad 2 \quad K \\ s^1 : \quad \frac{K}{2} - 1 \quad 0 \\ s^0 : \quad K \end{array}$$

For  $p$  to be stable, all entries in the 1st column must be positive:

$$K > 2 \quad \text{and} \quad K > 0 \quad (\text{already covered by } K > 1)$$

**Note:** The necessary condition requires  $K > 1$ , but now we actually know that we must have  $K > 2$  for stability.

## Some Comments on the Routh Test

- ▶ The result ( $\#(\text{RHP roots})$ ) is not affected if we multiply or divide any row of the Routh array by an arbitrary *positive* number.
- ▶ If we get a zero element in the 1st column, we can't continue. In that case, we can replace the 0 by a small number  $\varepsilon$  and apply Routh test to that. When we are done with the array, take the limit as  $\varepsilon \rightarrow 0$ . (see Ex. 3.33 in FPE)
- ▶ For an *entire row of zeros*, the procedure is a more complicated (see Example 3.34 in FPE) – we will not worry about this too much.