Plan of the Lecture

- Review: prototype 2nd-order system; transient response specifications
- Today's topic: system-modeling diagrams; interconnections; linearization

Goal: develop a methodology for representing and analyzing systems by means of block diagrams

System Modeling Diagrams

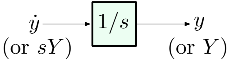
$$\text{large system} \underset{\text{compose}}{\overset{\text{decompose}}{\rightleftharpoons}} \text{smaller blocks (subsystems)}$$

— this is the core of systems theory

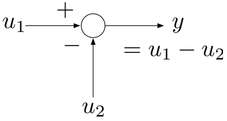
We will take smaller blocks from some given *library* and play with them to create/build more complicated systems.

All-Integrator Diagrams

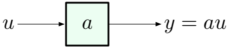
Our library will consist of three building blocks:



integrator



summing junction



constant gain

Two warnings:

- We can (and will) work either with u, y (time domain) or with U, Y (s-domain) — will often go back and forth
- When working with block diagrams, we typically ignore initial conditions.

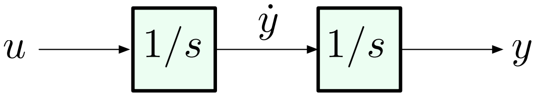
This is the *lowest level* we will go to in lectures; in the labs, you will implement these blocks using op amps.

Example 1

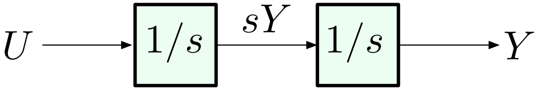
Build an all-integrator diagram for

$$\ddot{y} = u \qquad \Longleftrightarrow \qquad s^2 Y = U$$

This is obvious:



or



Example 2

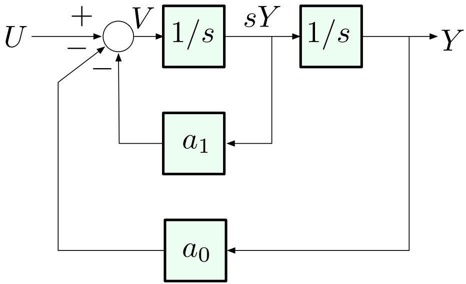
(building on Example 1)

$$\ddot{y} + a_1\dot{y} + a_0 y = u \qquad \Longleftrightarrow \qquad s^2 Y + a_1 sY + a_0 Y = U$$

$$\text{or} \qquad Y(s) = \frac{U(s)}{s^2 + a_1 s + a_0}$$

Always solve for the highest derivative:

$$\ddot{y} = \underbrace{-a_1\dot{y} - a_0 y + u}_{=v}$$

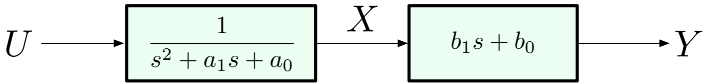


Example 3

Build an all-integrator diagram for a system with transfer function

$$H(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}$$

Step 1: decompose $H(s) = \dfrac{1}{s^2 + a_1 s + a_0} \cdot (b_1 s + b_0)$

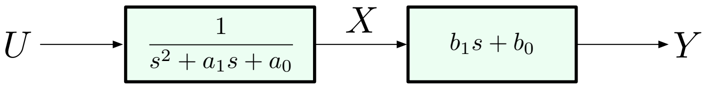


— here, X is an auxiliary (or intermediate) signal

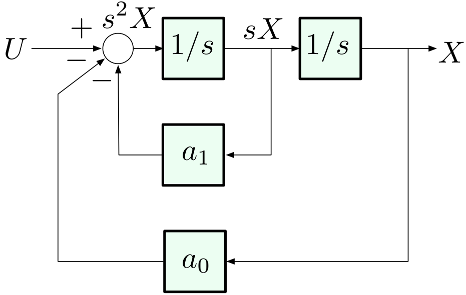
Note: $b_0 + b_1 s$ involves *differentiation*, which we cannot implement using an all-integrator diagram. But we will see that we don't need to do it directly.

Example 3, continued

Step 1: decompose $H(s) = \frac{1}{s^2 + a_1 s + a_0} \cdot (b_1 s + b_0)$



Step 2: The transformation $U \to X$ is from Example 2:

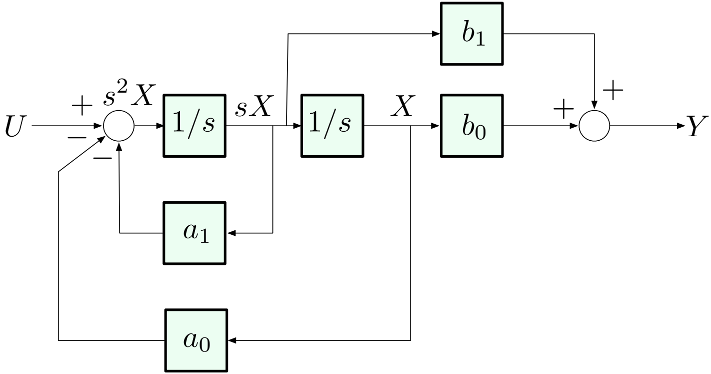


Example 3, continued

Step 3: now we notice that

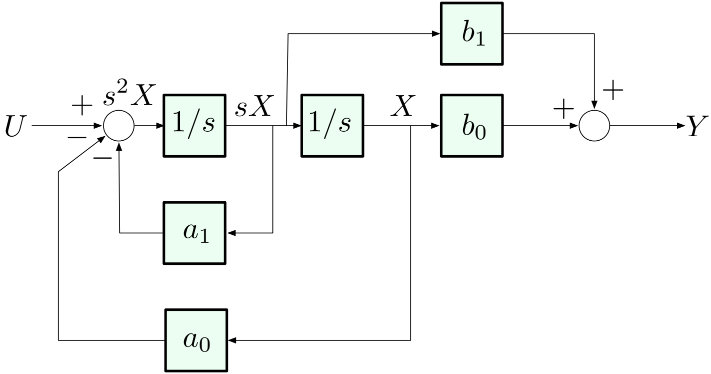
$$Y(s) = b_1 sX(s) + b_0 X(s),$$

and both X and sX are available signals in our diagram. So:



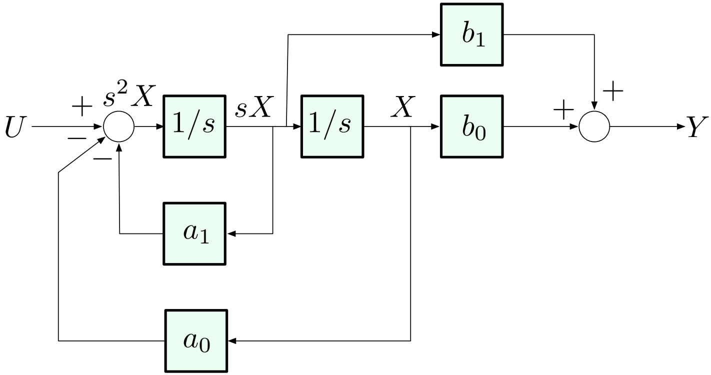
Example 3, continued

All-integrator diagram for $H(s) = \dfrac{b_1 s + b_0}{s^2 + a_1 s + a_0}$



Can we write down a state-space model corresponding to this diagram?

Example 3, continued



State-space model:

$$s^2X = U - a_1sX - a_0X \qquad Y = b_1sX + b_0X$$

$$\ddot{x} = -a_1\dot{x} - a_0x + u \qquad y = b_1\dot{x} + b_0x$$

Example 3, continued

State-space model:

$$\ddot{x} = -a_1\dot{x} - a_0 x + u \qquad y = b_1\dot{x} + b_0 x$$

$$x_1 = x,\ x_2 = \dot{x}$$

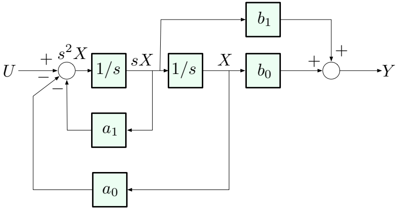
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \qquad y = \begin{pmatrix} b_0 & b_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

This is called *controller canonical form*.

- Easily generalizes to dimension > 1
- The reason behind the name will be made clear later in the semester

Example 3, wrap-up

All-integrator diagram for $H(s) = \dfrac{b_1 s + b_0}{s^2 + a_1 s + a_0}$



State-space model:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \qquad y = \begin{pmatrix} b_0 & b_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Important: for a given $H(s)$, the diagram is *not unique*. But, once we build a diagram, the state-space equations are unique (up to coordinate transformations).

Basic System Interconnections

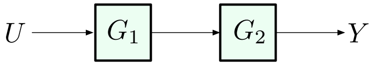
Now we will take this a level higher — we will talk about building complex systems from smaller blocks, without worrying about how those blocks look on the inside (they could themselves be all-integrator diagrams, etc.)

Block diagrams are an *abstraction* (they hide unnecessary "low-level" detail ...)

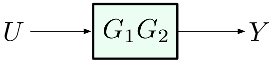
Block diagrams describe the *flow of information*

Basic System Interconnections: Series & Parallel

Series connection



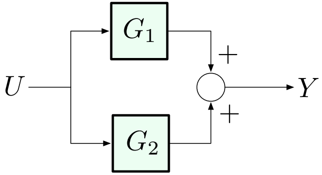
(G is common notation for t.f.'s)



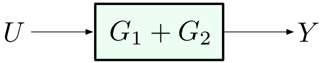
$$\frac{Y}{U} = G_1 G_2$$

(for SISO systems, the order of G_1 and G_2 does not matter)

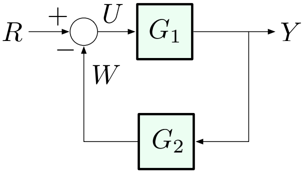
Parallel connection



$$\frac{Y}{U} = G_1 + G_2$$



Basic System Interconnections: Negative Feedback

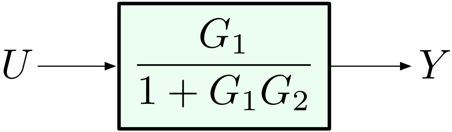


Find the transfer function from R (reference) to Y

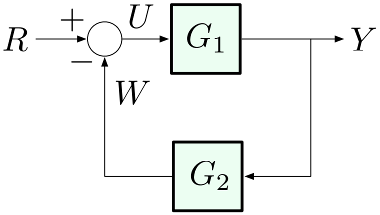
$$U = R - W$$

$$\begin{aligned} Y &= G_1 U \\ &= G_1(R - W) \\ &= G_1 R - G_1 G_2 Y \end{aligned}$$

$$\Longrightarrow \; Y = \frac{G_1}{1 + G_1 G_2} R$$



Basic System Interconnections: Negative Feedback



$$\implies Y = \frac{G_1}{1 + G_1 G_2} R$$

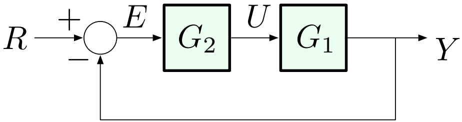
The gain of a negative feedback loop:

$$\frac{\text{forward gain}}{1 + \text{loop gain}}$$

This is an important relationship, easy to derive — no need to memorize it.

Unity Feedback

Other feedback configurations are also possible:

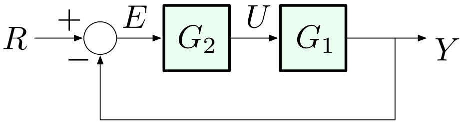


This is called *unity feedback* — no component on the feedback path.

Common structure (saw this in Lecture 1):

- R = reference
- U = control input
- Y = output
- E = error
- G_1 = plant (also denoted by P)
- G_2 = controller or compensator (also denoted by C or K)

Unity Feedback



Let's practice with deriving transfer functions: $\dfrac{\text{forward gain}}{1 + \text{loop gain}}$

- Reference R to output Y:

$$\frac{Y}{R} = \frac{G_1 G_2}{1 + G_1 G_2}$$

- Reference R to control input U:

$$\frac{U}{R} = \frac{G_2}{1 + G_1 G_2}$$

- Error E to output Y:

$$\frac{Y}{E} = G_1 G_2 \qquad \text{(no feedback path)}$$

Block Diagram Reduction

Given a complicated diagram involving series, parallel, and feedback interconnections, we often want to write down an overall transfer function from one of the variables to another.

This requires lots of practice: read FPE, Section 3.2 for examples.

General strategy:

- Name all the variables in the diagram
- Write down as many relationships between these variables as you can
- Learn to recognize series, parallel, and feedback interconnections
- Replace them by their equivalents
- Repeat

Review: State-Space Models

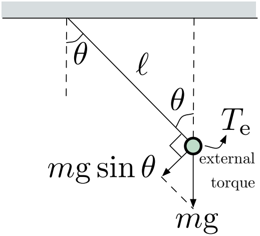
$$\dot{x} = Ax + Bu$$
$$y = Cx$$

State-space models are useful and convenient for writing down system models for different types of systems, in a unified manner.

What if the models are nonlinear?

— Linearization!

Example: Pendulum



Newton's 2nd law (rotational motion):

$$\underbrace{T}_{\text{total torque}} = \underbrace{J}_{\text{moment of inertia}} \underbrace{\alpha}_{\text{angular acceleration}}$$

$$= \text{pendulum torque} + \text{external torque}$$

$$\text{pendulum torque} = \underbrace{-mg\sin\theta}_{\text{force}} \cdot \underbrace{\ell}_{\text{lever arm}}$$

$$\text{moment of inertia } J = m\ell^2$$

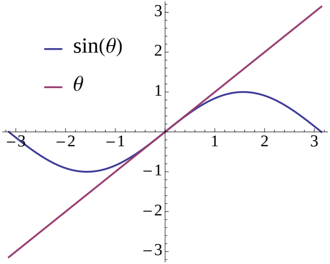
$$-mg\ell\sin\theta + T_e = m\ell^2\ddot{\theta}$$

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta + \frac{1}{m\ell^2}T_e \qquad \text{(nonlinear equation)}$$

Example: Pendulum

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta + \frac{1}{m\ell^2}T_e \qquad \text{(nonlinear equation)}$$

For *small* θ, use the approximation $\sin\theta \approx \theta$



$$\ddot{\theta} = -\frac{g}{\ell}\theta + \frac{1}{m\ell^2}T_e$$

State-space form: $\theta_1 = \theta,\ \theta_2 = \dot{\theta}$

$$\dot{\theta}_2 = -\frac{g}{\ell}\theta + \frac{1}{m\ell^2}T_e = -\frac{g}{\ell}\theta_1 + \frac{1}{m\ell^2}T_e$$

$$\begin{pmatrix}\dot{\theta}_1 \\ \dot{\theta}_2\end{pmatrix} = \begin{pmatrix}0 & 1 \\ -\frac{g}{\ell} & 0\end{pmatrix}\begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix} + \begin{pmatrix}0 \\ \frac{1}{m\ell^2}\end{pmatrix}T_e$$

Linearization

Taylor series expansion:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \dots$$

$$\approx f(x_0) + f'(x_0)(x - x_0) \qquad \text{linear approximation around } x = x_0$$

Control systems are generally *nonlinear*:

$$\dot{x} = f(x, u) \qquad \text{nonlinear state-space model}$$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \quad f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

Assume $x = 0, u = 0$ is an *equilibrium point*: $f(0, 0) = 0$

This means that, when the system is at rest and no control is applied, the system does not move.

Linearization

Linear approx. around $(x,u) = (0,0)$ to all components of f:

$$\dot{x}_1 = f_1(x,u), \qquad \ldots, \qquad \dot{x}_n = f_n(x,u)$$

For each $i = 1, \ldots, n$,

$$\begin{aligned} f_i(x,u) = \underbrace{f_i(0,0)}_{=0} &+ \frac{\partial f_i}{\partial x_1}(0,0)x_1 + \ldots + \frac{\partial f_i}{\partial x_n}(0,0)x_n \\ &+ \frac{\partial f_i}{\partial u_1}(0,0)u_1 + \ldots + \frac{\partial f_i}{\partial u_m}(0,0)u_m \end{aligned}$$

Linearized state-space model:

$$\dot{x} = Ax + Bu, \qquad \text{where } A_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\substack{x=0\\u=0}}, \ B_{ik} = \left.\frac{\partial f_i}{\partial u_k}\right|_{\substack{x=0\\u=0}}$$

Important: since we have ignored the higher-order terms, this linear system is only an *approximation* that holds only for *small deviations* from equilibrium.

Example: Pendulum, Revisited

Original nonlinear state-space model:

$$\dot{\theta}_1 = f_1(\theta_1, \theta_2, T_\mathrm{e}) = \theta_2 \qquad \text{— already linear}$$

$$\dot{\theta}_2 = f_2(\theta_1, \theta_2, T_\mathrm{e}) = -\frac{\mathrm{g}}{\ell}\sin\theta_1 + \frac{1}{m\ell^2}T_\mathrm{e}$$

Linear approx. of f_2 around equilibrium $(\theta_1, \theta_2, T_\mathrm{e}) = (0, 0, 0)$:

$$\frac{\partial f_2}{\partial \theta_1} = -\frac{\mathrm{g}}{\ell}\cos\theta_1 \qquad \frac{\partial f_2}{\partial \theta_2} = 0 \qquad \frac{\partial f_2}{\partial T_\mathrm{e}} = \frac{1}{m\ell^2}$$

$$\left.\frac{\partial f_2}{\partial \theta_1}\right|_0 = -\frac{\mathrm{g}}{\ell} \qquad \left.\frac{\partial f_2}{\partial \theta_2}\right|_0 = 0 \qquad \left.\frac{\partial f_2}{\partial T_\mathrm{e}}\right|_0 = \frac{1}{m\ell^2}$$

Linearized state-space model of the pendulum:

$$\dot{\theta}_1 = \theta_2$$

$$\dot{\theta}_2 = -\frac{\mathrm{g}}{\ell}\theta_1 + \frac{1}{m\ell^2}T_\mathrm{e}$$

valid for *small* deviations from equ.

General Linearization Procedure

- Start from nonlinear state-space model

$$\dot{x} = f(x, u)$$

- Find equilibrium point (x_0, u_0) such that $f(x_0, u_0) = 0$

  Note: different systems may have different equilibria, not necessarily $(0, 0)$, so we need to shift variables:

$$\underline{x} = x - x_0 \qquad \underline{u} = u - u_0$$
$$\underline{f}(\underline{x}, \underline{u}) = f(\underline{x} + x_0, \underline{u} + u_0) = f(x, u)$$

  Note that the transformation is *invertible*:

$$x = \underline{x} + x_0, \qquad u = \underline{u} + u_0$$

General Linearization Procedure

- Pass to shifted variables $\underline{x} = x - x_0$, $\underline{u} = u - u_0$

$$\begin{aligned} \dot{\underline{x}} &= \dot{x} \qquad\qquad (x_0 \text{ does not depend on } t) \\ &= f(x, u) \\ &= \underline{f}(\underline{x}, \underline{u}) \end{aligned}$$

— equivalent to original system

- The transformed system is in equilibrium at $(0, 0)$:

$$\underline{f}(0, 0) = f(x_0, u_0) = 0$$

- Now linearize:

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}, \qquad \text{where } A_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\substack{x=x_0 \\ u=u_0}}, \; B_{ik} = \left.\frac{\partial f_i}{\partial u_k}\right|_{\substack{x=x_0 \\ u=u_0}}$$

General Linearization Procedure

- Why do need the shift $\underline{x} = x - x_0, \underline{u} = u - u_9$?
- This requires some thought. Indeed, we may talk about a *linear approximation* of any smooth function f at any point x_0:

$$f(x) \approx f(x_0)+f'(x_0)(x-x_0) \qquad \text{— } f(x_0) \text{ does not have to be } 0$$

- The key is that we want to approximate a given nonlinear system $\dot{x} = f(x, u)$ by a *linear* system $\dot{x} = Ax + Bu$ (may have to shift coordinates: $x \mapsto x - x_0$, $u \mapsto u - u_0$)

  Any linear system *must* have an equilibrium point at $(x, u) = (0, 0)$:

$$f(x, u) = Ax + Bu \qquad f(0, 0) = A0 + B0 = 0.$$