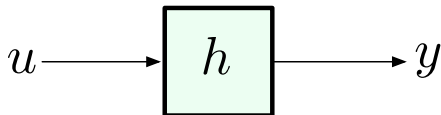


Plan of the Lecture

- ▶ **Review:** dynamic response; transfer functions; transient and steady-state response
- ▶ **Today's topic:** dynamic response (transient and steady-state) with arbitrary I.C.'s

Goal: develop a methodology for characterizing the output of a given system for a given input.

Dynamic Response



Problem: compute the response y to a given input u under a given set of initial conditions.

In particular, we wish to know both the **transient response** (due to I.C.'s) and the **steady-state response** (once the effect of the I.C.'s “washes away”).

Laplace Transforms Revisited

(see FPE, Appendix A)

One-sided (or unilateral) Laplace transform:

$$\mathcal{L}\{f(t)\} \equiv F(s) = \int_0^{\infty} f(t)e^{-st} dt \quad (\text{really, from } 0^-)$$

— for simple functions f , can compute $\mathcal{L}f$ by hand.

Example: unit step

$$f(t) = 1(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\mathcal{L}\{1(t)\} = \int_0^{\infty} e^{-st} dt = -\frac{1}{s}e^{-st} \Big|_0^{\infty} = \frac{1}{s} \quad (\text{pole at } s = 0)$$

— this is valid provided $\text{Re}(s) > 0$, so that $e^{-st} \xrightarrow{t \rightarrow +\infty} 0$.

Laplace Transforms Revisited

Example: $f(t) = \cos t$

$$\mathcal{L}\{\cos t\} = \mathcal{L}\left\{\frac{1}{2}e^{jt} + \frac{1}{2}e^{-jt}\right\} \quad (\text{Euler's formula})$$

$$= \frac{1}{2}\mathcal{L}\{e^{jt}\} + \frac{1}{2}\mathcal{L}\{e^{-jt}\} \quad (\text{linearity})$$

$$\mathcal{L}\{e^{jt}\} = \int_0^{\infty} e^{jt} e^{-st} dt = \int_0^{\infty} e^{(j-s)t} dt = \frac{1}{j-s} e^{(j-s)t} \Big|_0^{\infty}$$

$$= -\frac{1}{j-s} \quad (\text{pole at } s = j)$$

$$\mathcal{L}\{e^{-jt}\} = \int_0^{\infty} e^{-jt} e^{-st} dt = \int_0^{\infty} e^{-(j+s)t} dt = -\frac{1}{j+s} e^{-(j+s)t} \Big|_0^{\infty}$$

$$= \frac{1}{j+s} \quad (\text{pole at } s = -j)$$

— in both cases, require $\text{Re}(s) > 0$, i.e., s must lie in the right half-plane (RHP)

Laplace Transforms Revisited

Example: $f(t) = \cos t$

$$\begin{aligned}\mathcal{L}\{\cos t\} &= \frac{1}{2}\mathcal{L}\{e^{jt}\} + \frac{1}{2}\mathcal{L}\{e^{-jt}\} \\ &= \frac{1}{2}\left(-\frac{1}{j-s} + \frac{1}{j+s}\right) \\ &= \frac{1}{2}\left(\frac{-j-s+j-s}{(j-s)(j+s)}\right) \\ &= \frac{1}{2}\left(\frac{-2s}{-1 + \cancel{js} - \cancel{js} - s^2}\right) \\ &= \frac{s}{s^2 + 1} \quad (\text{poles at } s = \pm j)\end{aligned}$$

for $\text{Re}(s) > 0$

Laplace Transforms Revisited

Convolution: $\mathcal{L}\{f \star g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$
(useful because $Y(s) = H(s)U(s)$)

Example: $\dot{y} = -y + u \quad y(0) = 0$

Compute the response for $u(t) = \cos t$

We already know

$$H(s) = \frac{1}{s+1} \quad (\text{from earlier example})$$

$$U(s) = \frac{s}{s^2+1} \quad (\text{just proved})$$

$$\implies Y(s) = H(s)U(s) = \frac{s}{(s+1)(s^2+1)}$$

$$y(t) = \mathcal{L}^{-1}\{Y\}$$

— can't find $Y(s)$ in the tables. So how do we compute y ?

Method of Partial Fractions

Problem: compute $\mathcal{L}^{-1} \left\{ \frac{s}{(s+1)(s^2+1)} \right\}$

This Laplace transform is not in the tables, but let's look at the table anyway. What do we find?

$$\frac{1}{s+1} \quad \mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\} = e^{-t} \quad (\#7)$$

$$\frac{1}{s^2+1} \quad \mathcal{L}^{-1} \left\{ \frac{1}{s^2+1} \right\} = \sin t \quad (\#17)$$

$$\frac{s}{s^2+1} \quad \mathcal{L}^{-1} \left\{ \frac{s}{s^2+1} \right\} = \cos t \quad (\#18)$$

— so we see some things that are similar to $Y(s)$, but not quite.

This brings us to the [method of partial fractions](#):

- ▶ boring (i.e., character-building), but *very useful*
- ▶ allows us to break up complicated fractions into sums of simpler ones, for which we know \mathcal{L}^{-1} from tables

Method of Partial Fractions

Problem: compute $\mathcal{L}^{-1}\{Y(s)\}$, where

$$Y(s) = \frac{s}{(s+1)(s^2+1)}$$

We seek a, b, c , such that

$$Y(s) = \frac{a}{s+1} + \frac{bs+c}{s^2+1} \quad (\text{need } bs+c \text{ so that } \deg(\text{num}) = \deg(\text{den}) - 1)$$

► Find a : multiply by $s+1$ to isolate a

$$(s+1)Y(s) = \frac{s}{s^2+1} = a + \frac{(s+1)(as+b)}{(s^2+1)}$$

— now let $s = -1$ to “kill” the second term on the RHS:

$$a = (s+1)Y(s) \Big|_{s=-1} = -\frac{1}{2}$$

Method of Partial Fractions

Problem: compute $\mathcal{L}^{-1}\{Y(s)\}$, where

$$Y(s) = \frac{s}{(s+1)(s^2+1)}$$

We seek a, b, c , such that

$$Y(s) = \frac{a}{s+1} + \frac{bs+c}{s^2+1} \quad (\text{need } bs+c \text{ so that } \deg(\text{num}) = \deg(\text{den}) - 1)$$

► Find b : multiply by s^2+1 to isolate $bs+c$

$$(s^2+1)Y(s) = \frac{s}{s+1} = \frac{a(s^2+1)}{s+1} + bs+c$$

— now let $s=j$ to “kill” the first term on the RHS:

$$bj+c = (s^2+1)Y(s) \Big|_{s=j} = \frac{j}{1+j}$$

Match $\text{Re}(\cdot)$ and $\text{Im}(\cdot)$ parts:

$$c+bj = \frac{j}{1+j} = \frac{j(1-j)}{(1+j)(1-j)} = \frac{1}{2} + \frac{j}{2} \implies b=c = \frac{1}{2}$$

Method of Partial Fractions

Problem: compute $\mathcal{L}^{-1}\{Y(s)\}$, where

$$Y(s) = \frac{s}{(s+1)(s^2+1)}$$

We found that

$$Y(s) = -\frac{1}{2(s+1)} + \frac{s}{2(s^2+1)} + \frac{1}{2(s^2+1)}$$

Now we can use linearity and tables:

$$\begin{aligned}y(t) &= \mathcal{L}^{-1} \left\{ -\frac{1}{2(s+1)} + \frac{s}{2(s^2+1)} + \frac{1}{2(s^2+1)} \right\} \\&= -\frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\} + \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{s}{s^2+1} \right\} + \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{1}{s^2+1} \right\} \\&= -\frac{1}{2} e^{-t} + \frac{1}{2} \cos t + \frac{1}{2} \sin t \quad (\text{from tables}) \\&= -\frac{1}{2} e^{-t} + \frac{1}{\sqrt{2}} \cos(t - \pi/4) \quad (\cos(a-b) = \cos a \cos b + \sin a \sin b)\end{aligned}$$

Transient and Steady-State Response

Consider the system $\dot{y} = -y + u$ $y(0) = 0$

$$u(t) = \cos t \quad \longrightarrow \quad y(t) = \underbrace{-\frac{1}{2}e^{-t}}_{\text{transient response}} + \underbrace{\frac{1}{\sqrt{2}} \cos(t - \pi/4)}_{\text{steady-state response}}$$

— transient response vanishes as $t \rightarrow \infty$ (we will see later why)

Let's compare against the frequency response formula:

$$H(s) = \frac{1}{s+1} \quad \Longrightarrow \quad H(j\omega) = \frac{1}{j\omega+1}$$

$u(t) = \cos t$ has $A = 1$ and $\omega = 1$, so

$$\begin{aligned} y(t) &= M(1) \cos(t + \varphi(1)) \\ &= \frac{1}{\sqrt{2}} \cos(t - \pi/4) \end{aligned}$$

— the freq. response formula gives only the steady-state part!!

Transient and Steady-State Response

Consider the system $\dot{y} = -y + u$ $y(0) = 0$

We computed the response to $u(t) = \cos t$ in two ways:

$$y(t) = -\frac{1}{2}e^{-t} + \frac{1}{\sqrt{2}} \cos(t - \pi/4)$$

— using the method of partial fractions;

$$y(t) = \frac{1}{\sqrt{2}} \cos(t - \pi/4)$$

— using the frequency response formula.

Q: Which answer is correct? And why?

A: At $t = 0$, $\frac{1}{\sqrt{2}} \cos(t - \pi/4) = \frac{1}{2} \neq 0$, which is inconsistent

with the initial condition $y(0) = 0$. The term $-\frac{1}{2}e^{-t} \Big|_{t=0} = -\frac{1}{2}$ cancels the steady-state term, so indeed $y(0) = 0$.

Therefore, the first formula is correct.

Transient and Steady-State Response

Main message: the frequency response formula only gives the steady-state part of the response, but the inverse Laplace transform gives the whole response (including the transient part).

— we will now see how to deal with nonzero I.C.'s ...

Laplace Transforms and Differentiation

Given a differentiable function f , what is the Laplace transform $\mathcal{L}\{f'(t)\}$ of its time derivative?

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= \int_0^{\infty} f'(t)e^{-st} dt \\ &= f(t)e^{-st} \Big|_0^{\infty} + s \int_0^{\infty} e^{-st} f(t) dt \quad (\text{integrate by parts}) \\ &= -f(0) + sF(s)\end{aligned}$$

— provided $f(t)e^{-st} \rightarrow 0$ as $t \rightarrow \infty$

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0) \quad \text{— this is how we account for I.C.'s}$$

Similarly:

$$\begin{aligned}\mathcal{L}\{f''(t)\} &= \mathcal{L}\{(f'(t))'\} = s\mathcal{L}\{f'(t)\} - f'(0) \\ &= s^2F(s) - sf(0) - f'(0)\end{aligned}$$

Example

Consider the system

$$\ddot{y} + 3\dot{y} + 2y = u, \quad y(0) = \dot{y}(0) = 0$$

(need two I.C.'s for 2nd-order ODE's)

Let's compute the transfer function: $H(s) = \frac{Y(s)}{U(s)}$

— take Laplace transform of both sides (zero I.C.'s):

$$s^2Y(s) + 3sY(s) + 2Y(s) = U(s) \quad H(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + 3s + 2}$$

Example (continued)

$$\ddot{y} + 3\dot{y} + 2y = u, \quad y(0) = \alpha, \dot{y}(0) = \beta$$

Compute the *step response*, i.e., response to $u(t) = 1(t)$

Caution!! $Y(s) = H(s)U(s)$ no longer holds if $\alpha \neq 0$ or $\beta \neq 0$

Again, take Laplace transforms of both sides, mind the I.C.'s:

$$s^2Y(s) - s\alpha - \beta + 3sY(s) - 3\alpha + 2Y(s) = U(s)$$

$U(s) = \mathcal{L}\{1(t)\} = 1/s$, which gives

$$s^2Y(s) - s\alpha - \beta + 3sY(s) - 3\alpha + 2Y(s) = \frac{1}{s}$$

$$Y(s) = \frac{\alpha s + (3\alpha + \beta) + \frac{1}{s}}{s^2 + 3s + 2} = \frac{\alpha s^2 + (3\alpha + \beta)s + 1}{s(s+1)(s+2)}$$

Note: if $\alpha = \beta = 0$, then $Y(s) = \frac{1}{s(s+1)(s+2)} = H(s)U(s)$

Example (continued)

Compute the step response of

$$\ddot{y} + 3\dot{y} + 2y = u, \quad y(0) = \alpha, \dot{y}(0) = \beta$$

$$Y(s) = \frac{\alpha s^2 + (3\alpha + \beta)s + 1}{s(s+1)(s+2)} \quad y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

Use the method of partial fractions:

$$\frac{\alpha s^2 + (3\alpha + \beta)s + 1}{s(s+1)(s+2)} = \frac{a}{s} + \frac{b}{s+1} + \frac{c}{s+2}$$

— this gives $a = 1/2$, $b = 2\alpha + \beta - 1$, $c = -\alpha - \beta + 1/2$

$$Y(s) = \frac{1}{2s} + (2\alpha + \beta - 1)\frac{1}{s+1} + \frac{-\alpha - \beta + 1/2}{s+2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{2}1(t) + (2\alpha + \beta - 1)e^{-t} + (1/2 - \alpha - \beta)e^{-2t}$$

Example (continued)

The step response of

$$\ddot{y} + 3\dot{y} + 2y = u, \quad y(0) = \alpha, \dot{y}(0) = \beta$$

is given by

$$y(t) = \frac{1}{2}1(t) + (2\alpha + \beta - 1)e^{-t} + (1/2 - \alpha - \beta)e^{-2t}$$

What are the transient and the steady-state terms?

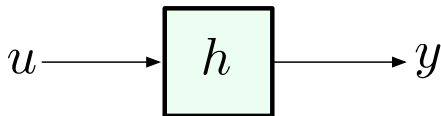
- ▶ The transient terms are e^{-t} , e^{-2t} (decay to zero at exponential rates -1 and -2)

Note the poles of $H(s) = \frac{1}{(s+1)(s+2)}$ at $s = -1$ and $s = -2$

— these are *stable poles* (both lie in LHP)

- ▶ the steady-state part is $\frac{1}{2}1(t)$ — converges to steady-state value of $1/2$

DC Gain



Definition: the steady-state value of the step response is called the *DC gain* of the system.

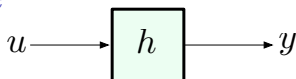
$$\text{DC gain} = y(\infty) = \lim_{t \rightarrow \infty} y(t) \quad \text{for } u(t) = 1(t)$$

In our example above, the step response is

$$y(t) = \frac{1}{2}1(t) + (2\alpha + \beta - 1)e^{-t} + (1/2 - \alpha - \beta)e^{-2t}$$

therefore, DC gain = $y(\infty) = 1/2$

Steady-State Value



$$u(t) = 1(t) \quad U(s) = \frac{1}{s} \quad \Longrightarrow \quad Y(s) = \frac{H(s)}{s}$$

— can we compute $y(\infty)$ from $Y(s)$?

Let's look at some examples:

▶ $Y(s) = \frac{1}{s+a}, a > 0$ (pole at $s = -a < 0$)
 $y(t) = e^{-at} \implies y(\infty) = 0$

▶ $Y(s) = \frac{1}{s+a}, a < 0$ (pole at $s = -a > 0$)
 $y(t) = e^{-at} \implies y(\infty) = \infty$

▶ $Y(s) = \frac{1}{s^2 + \omega^2}, \omega \in \mathbb{R}$ (poles at $s = \pm j\omega$, purely imaginary)
 $y(t) = \sin(\omega t) \implies y(\infty)$ does not exist

▶ $Y(s) = \frac{c}{s}$ (pole at the origin, $s = 0$)
 $y(t) = c1(t) \implies y(\infty) = c$

The Final Value Theorem

We can now deduce the **Final Value Theorem (FVT)**:

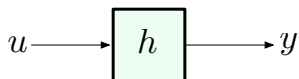
If all poles of $sY(s)$ are *strictly stable* or lie in the *open left half-plane* (OLHP), i.e., have $\text{Re}(s) < 0$, then

$$y(\infty) = \lim_{s \rightarrow 0} sY(s).$$

In our examples, multiply $Y(s)$ by s , check poles:

- ▶ $Y(s) = \frac{1}{s+a}$ $sY(s) = \frac{s}{s+a}$
if $a > 0$, then $y(\infty) = 0$; if $a < 0$, FVT does not give correct answer
- ▶ $Y(s) = \frac{1}{s^2 + \omega^2}$ $sY(s) = \frac{s}{s^2 + \omega^2}$
poles are purely imaginary (not in OLHP), FVT does not give correct answer
- ▶ $Y(s) = \frac{c}{s}$ $sY(s) = c$
poles at infinity, so $y(\infty) = c$ – FVT gives correct answer

Back to DC Gain



Step response: $Y(s) = \frac{H(s)}{s}$

— if all poles of $sY(s) = H(s)$ are strictly stable, then

$$y(\infty) = \lim_{s \rightarrow 0} H(s)$$

by the FVT.

Example: compute DC gain of the system with transfer function

$$H(s) = \frac{s^2 + 5s + 3}{s^3 + 4s + 2s + 5}$$

All poles of $H(s)$ are strictly stable (we will see this later using the *Routh–Hurwitz criterion*), so

$$y(\infty) = H(s) \Big|_{s=0} = \frac{3}{5}.$$