ECE 486: Control Systems

Lecture 6B: Stability

Key Takeaways

We study the properties exponential terms e^{st} that appear in the free and forced response.

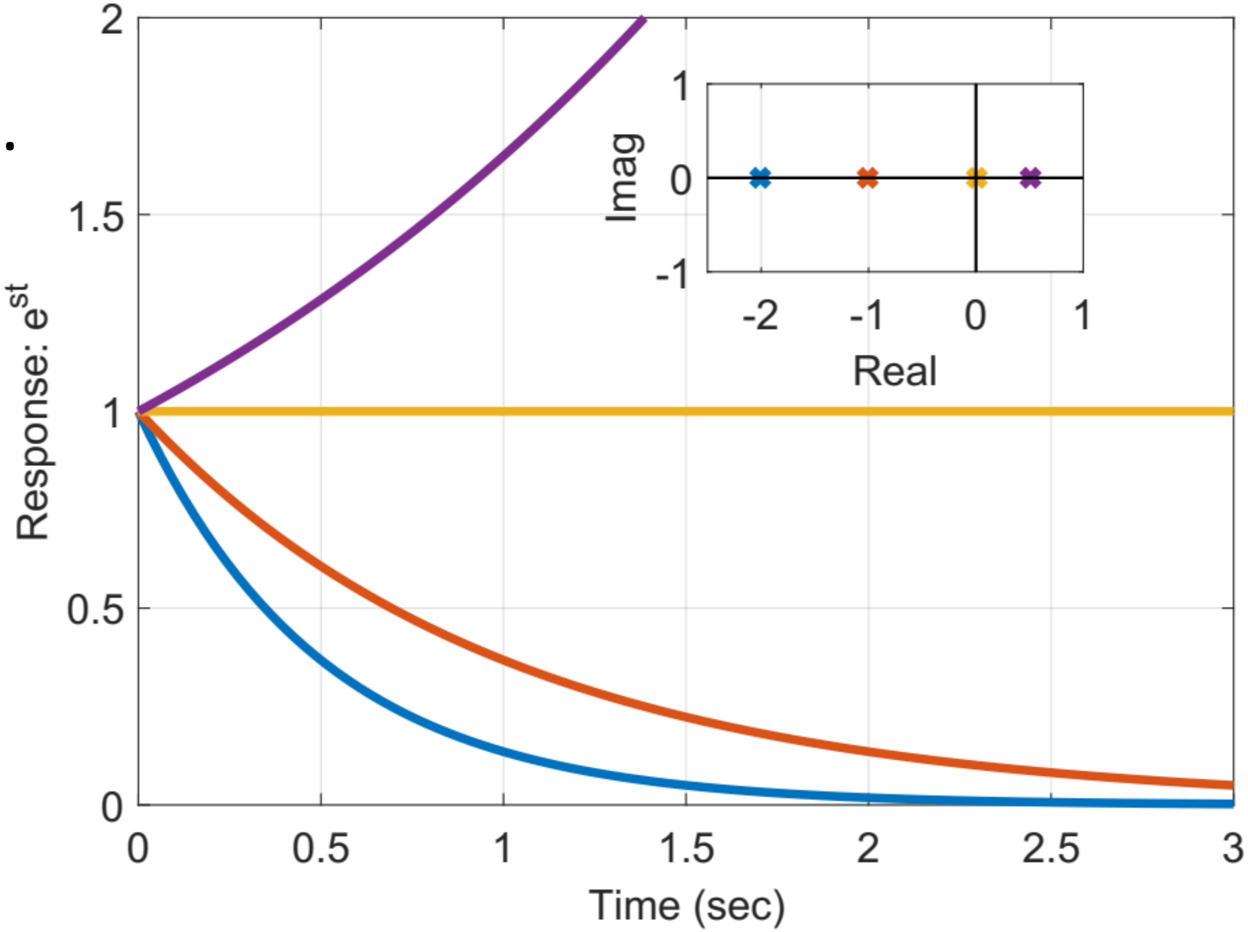
The lecture covers the following:

1. Response characteristics for real and complex roots
2. Time Constants
3. Internal Stability
4. Bounded-Input, Bounded-Output Stability

Exponential Response: Real Root

Key properties of e^{st} with a real root s.

- s≥0: Response stays constant or grows unbounded.
- s<0: Response decays to zero.
- Faster decay for more negative values of s.



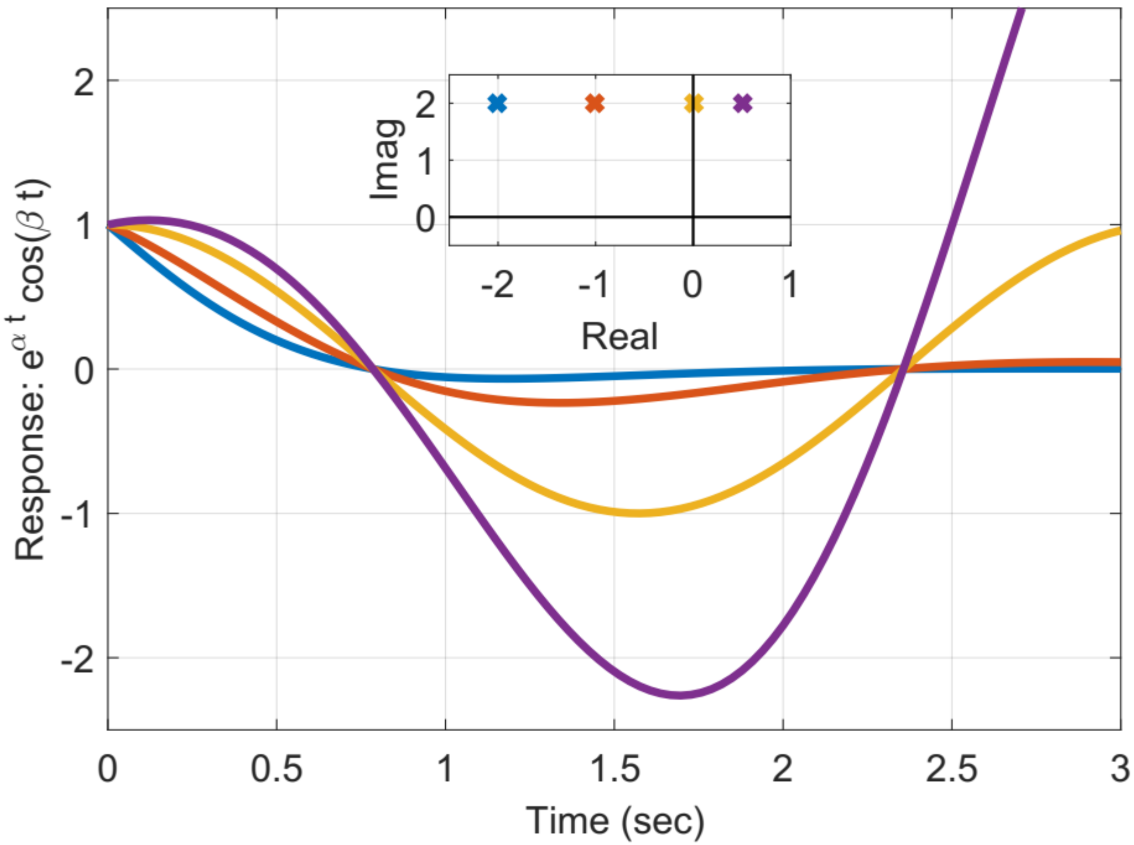
Exponential Response: Complex Roots

Rewrite e^{st} with a complex roots $s = \alpha \pm j\beta$ as:

$$e^{\alpha t}\cos(\beta t) \text{ and } e^{\alpha t}\sin(\beta t)$$

Key Properties

- Response oscillates
- $\alpha \geq 0$: Amplitude stays constant or grows unbounded.
- $\alpha < 0$: Amplitude decays to zero.
- Faster decay for more negative values of α.



Summary of Key Properties

Terminology

- The left half of the complex plane (LHP) corresponds to values of *s* with *Re(s) < 0*.
- The closed right half of the complex plane (CRHP) corresponds to values of *s* with *Re(s) ≥ 0*.
- The time constant of a pole $s \in C$ is $\tau = \frac{1}{|Re(s)|}$ sec.

Important Facts:

1. **The exponential term decays to zero if and only if s is in the LHP.**
2. **If *s* is in the LHP then the exponential term decays to 0.05 (=5% of initial value) in 3τ seconds.**

(Reason: $e^{st}|_{t=3\tau} = e^{-3} \approx 0.05$)

Internal Stability

- An LTI system is internally stable if the free response returns to zero ($y(t) \to 0$ as $t \to \infty$) for any initial condition. It is internally unstable if it is not stable.

- **Fact:** A linear system is internally stable if and only if all poles are in the LHP, i.e. $\mathrm{Re}(s_i) < 0$ for all i.

- **Reason:** Free response solution has the form:

$$y(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} + \cdots + c_n e^{s_n t}$$

Input-Output Stability

- An LTI system is bounded-input, bounded-output (BIBO) stable if the forced response output with zero ICs remains bounded for every bounded input. The system is BIBO unstable if it is not BIBO stable.
- **Fact:** A minimal, linear system is BIBO stable if and only if all poles are in the LHP.
- **Reason:** More technical and will come later.
  - The fact assumes system is minimal as pole/zero cancellations can "hide" unstable dynamics.

    Example: $\dot{y}(t) - y(t) = \dot{u}(t) - u(t) \text{ and } G(s) = \frac{s-1}{s-1}$

For minimal LTI systems the two types of stability are equivalent. We will often not distinguish between them.

Routh-Hurwitz Condition

A minimal, LTI system is stable if and only all poles are in the LHP. We can numerically compute all poles:

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

The Routh-Hurwitz provides necessary and sufficient conditions on $\{a_0, a_1, \ldots, a_n\}$ for all roots to lie in the LHP.

Special Cases:

System Order	Characteristic Eqn	Routh-Hurwitz Conditions
First	$s+a_0=0$	$a_0>0$
Second	$s^2+a_1s+a_0=0$	$a_0>0$ and $a_1>0$
Third	$s^3+a_2s^2+a_1s+a_0=0$	$a_2>0$, $a_0>0$, and $a_1 a_2 > a_0$