ECE 486: Control Systems

Lecture 4C: Second-Order Step Response

Key Takeaways

This lecture covers the step response for second-order systems.

The step response of a *stable*, second-order system.

1. Is characterized by the natural frequency and damping ratio of the system
2. Has overshoot and oscillations if the system is underdamped.

Second-Order Step Response

Consider the second-order system:

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = b_0 u(t)$$

$$\text{with } y(0) = 0,\ \dot{y}(0) = 0$$

$$u(t) = 1 \text{ for all } t \geq 0$$

$$G(s) = \frac{b_0}{s^2 + a_1 s + a_0}$$

The system is stable (both roots in LHP) if and only if a_1, $a_0 > 0$.

For stable systems, the coefficients are typically redefined:

$$\ddot{y}(t) + \underbrace{2\zeta\omega_n}_{=a_1}\dot{y}(t) + \underbrace{\omega_n^2}_{=a_0} y(t) = b_0 u(t)$$

where:

- ω_n := Natural frequency (rad/sec)
- ζ := Damping ratio (unitless)

Overdamped/Underdamped Systems

Stable, second-order system:

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = b_0 u(t) \qquad G(s) = \frac{b_0}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Two poles are given by:

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Three cases depending on $\zeta^2 - 1$:

- Overdamped, $\zeta \geq 1$: Roots are real and distinct
- Critically Damped, $\zeta = 1$: Roots are real and both at $s_{1,2} = -\zeta\omega_n$
- Underdamped, $\zeta < 1$: Roots are a complex conjugate pair.

Overdamped/Underdamped Systems

Stable, second-order system:

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = b_0 u(t) \qquad G(s) = \frac{b_0}{s^2+2\zeta\omega_n s+\omega_n^2}$$

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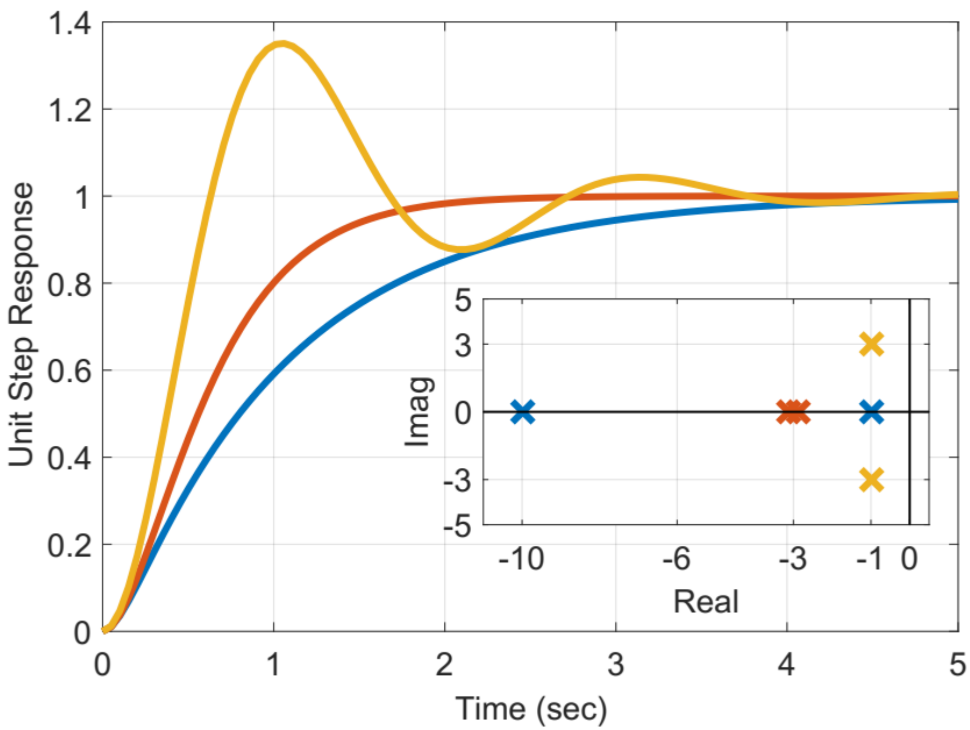


Over/Critically damped solutions (red/blue) are similar to first-order response.

Underdamped solution (yellow) has overshoot and oscillations.

Underdamped Poles

If $\zeta < 1$ then the poles are:

$$s_{1,2} = -\zeta\omega_n \pm j\,\underbrace{\omega_n\sqrt{1-\zeta^2}}_{:=\omega_d}$$

Imaginary part ω_d is called the the damped natural frequency.

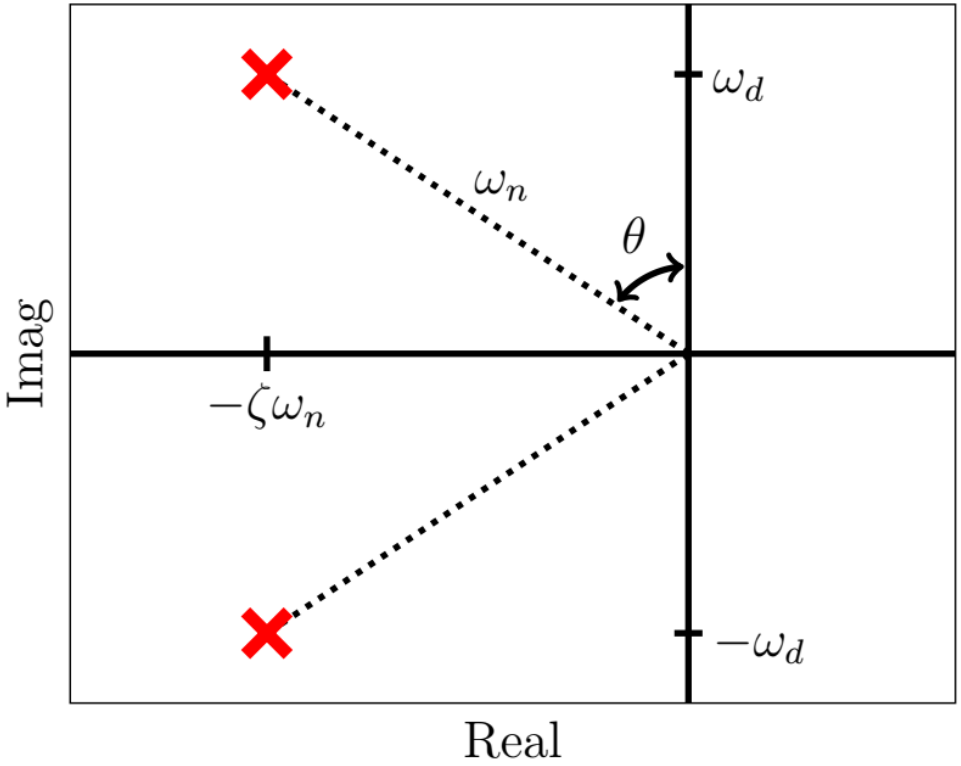
Time constant is:

$$\tau = \frac{1}{\zeta\omega_n}$$

Angle θ is given by:

$$\sin(\theta) = \zeta$$

Angle decreases for smaller values of ζ.



Underdamped Poles

If $\zeta < 1$ then the poles are:

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

Example:

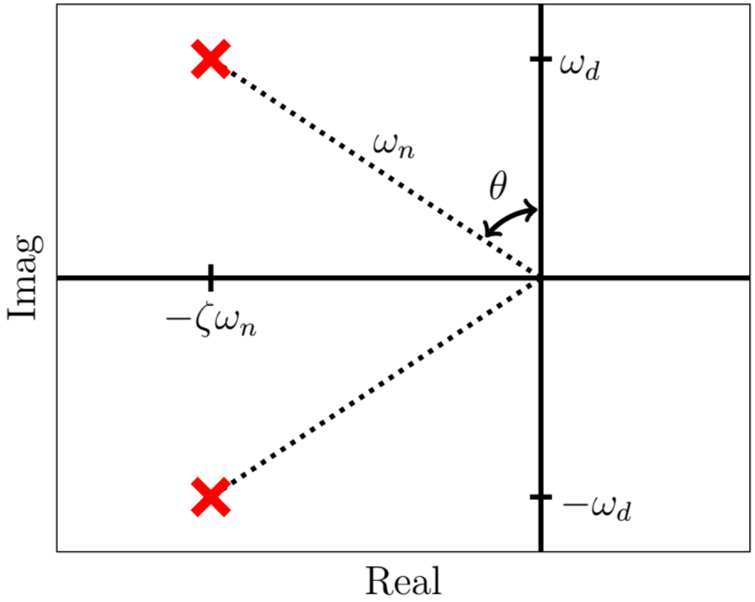
$$\ddot{y}(t) + 2\dot{y}(t) + 10y(t) = 10u(t) \qquad G(s) = \frac{10}{s^2+2s+10}$$

$$\omega_n^2 = 10 \Rightarrow \omega_n = \sqrt{10} \approx 3.2\frac{rad}{sec}$$

$$2\zeta\omega_n = 2 \Rightarrow \zeta = \frac{2}{2\sqrt{10}} \approx 0.32$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 3\frac{rad}{sec}$$

$$s_{1,2} = -1 \pm 3j$$



Underdamped Poles

If $\zeta < 1$ then the poles are:

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

Example:

$$\ddot{y}(t) + 2\dot{y}(t) + 10y(t) + 10u(t) \qquad G(s) = \frac{10}{s^2+2s+10}$$

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>> G = tf(10,[1 2 10]);

% Display poles, damping ratio, nat. freqs., time
constants
>> damp(G)
         Pole                Damping   Frequency    Time Constant
                                     (rad/seconds)  (seconds)
-1.00e+00 + 3.00e+00i   3.16e-01    3.16e+00      1.00e+00
-1.00e+00 - 3.00e+00i   3.16e-01    3.16e+00      1.00e+00
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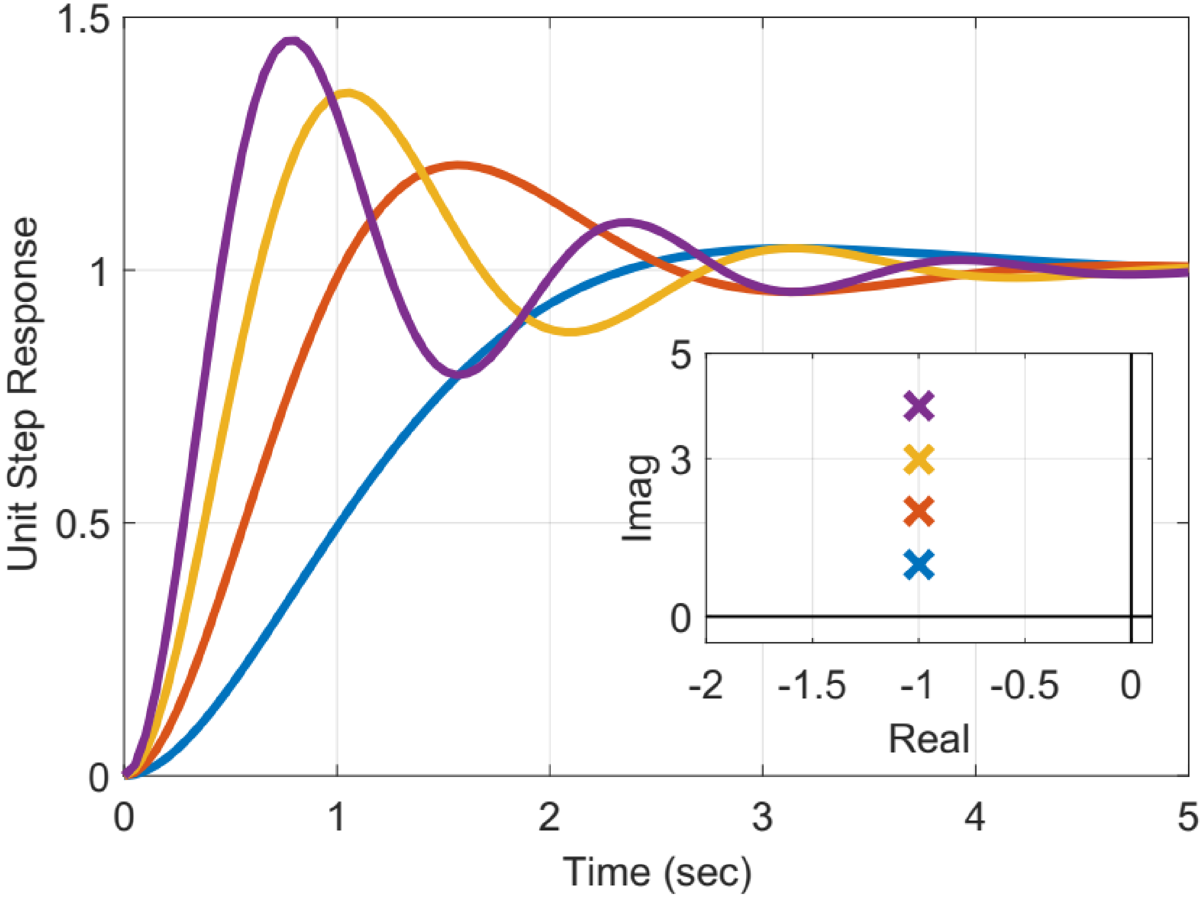
Key Features: Stable, Underdamped Step Response

$$y(t) = \bar{y}\left(1 - e^{-\zeta\omega_n t}\cos(\omega_d t) - e^{-\zeta\omega_n t}\frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right) \quad \text{where } \bar{y} := \frac{b_0}{\omega_n^2}$$

- Final Value: $\bar{y} = G(0)\bar{u}$
- Settling Time: $T_s = \frac{3}{\zeta\omega_n}$
- Peak Overshoot: $M_p = e^{-\frac{\zeta}{\sqrt{1-\zeta^2}}\pi}$ where $M_p = \frac{y(T_p)-\bar{y}}{\bar{y}}$
- Peak Time: $T_p = \frac{\pi}{\omega_d}$
- Rise Time: $T_r \approx \frac{1.8}{\omega_n}$

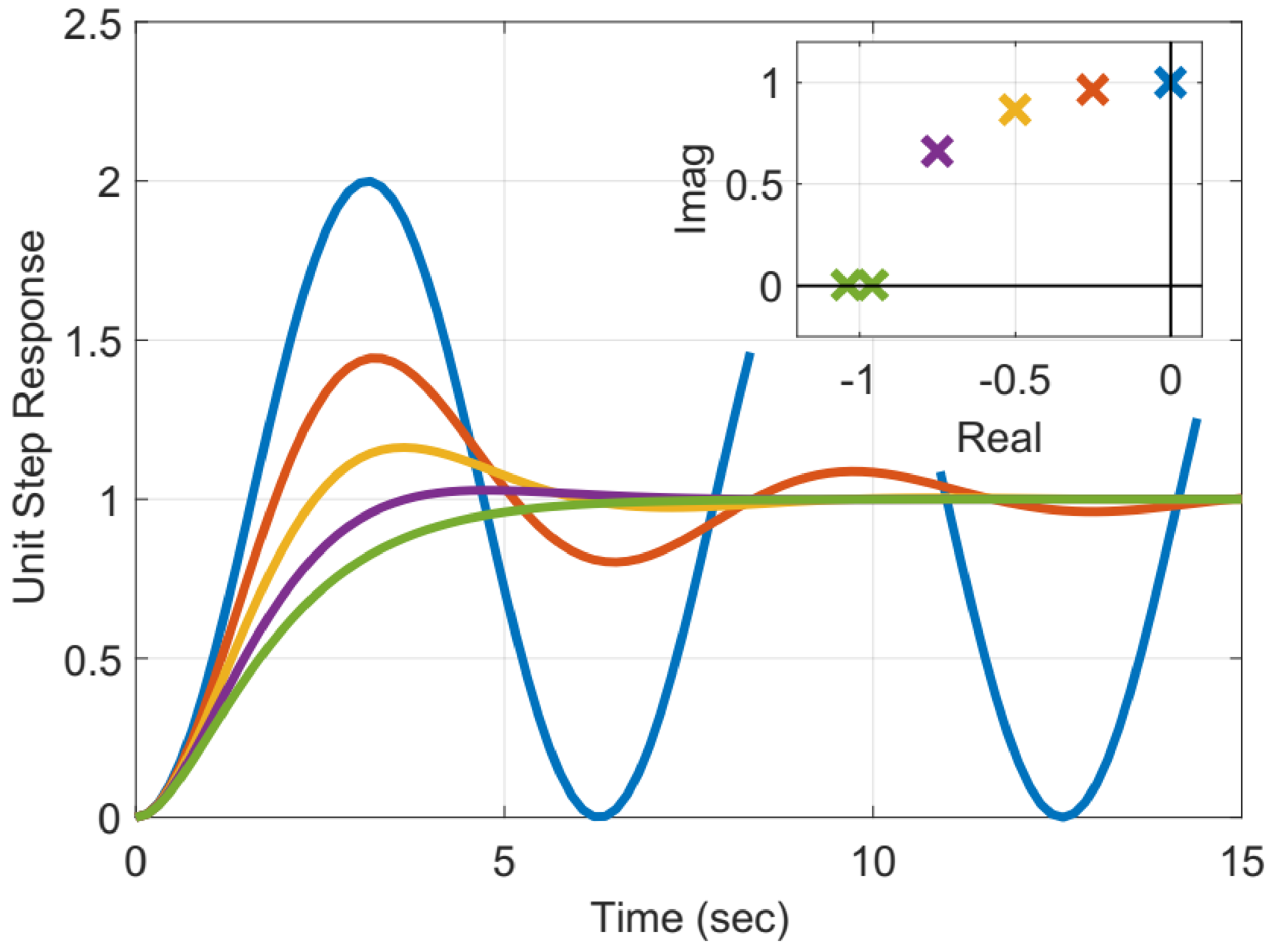
Underdamped Step Response

$$y(t) = \bar{y}\left(1 - e^{-\zeta\omega_n t}\cos(\omega_d t) - e^{-\zeta\omega_n t}\frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right) \text{ where } \bar{y} := \frac{b_0}{\omega_n^2}$$



Underdamped Step Response

$$y(t) = \bar{y}\left(1 - e^{-\zeta\omega_n t}\cos(\omega_d t) - e^{-\zeta\omega_n t}\frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right) \quad \text{where } \bar{y} := \frac{b_0}{\omega_n^2}$$



TD Specs in Frequency Domain

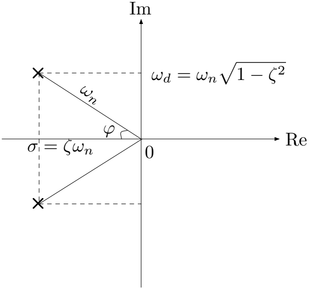
We want to *visualize* time-domain specs in terms of *admissible pole locations* for the 2nd-order system

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\sigma^2 + \omega_d^2}{(s+\sigma)^2 + \omega_d^2}$$

where $\sigma = \zeta\omega_n$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Step response: $y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$



$$\omega_n^2 = \sigma^2 + \omega_d^2$$

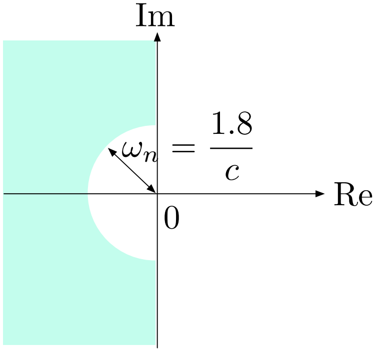
$$\zeta = \cos\varphi$$

Rise Time in Frequency Domain

Suppose we want $t_r \leq c$ (c is some desired given value)

$$t_r \approx \frac{1.8}{\omega_n} \leq c \qquad \Longrightarrow \qquad \omega_n \geq \frac{1.8}{c}$$

Geometrically, we want poles to lie in the shaded region:



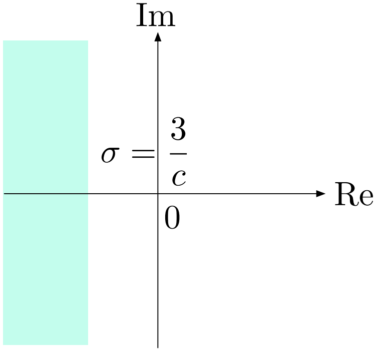
(recall that ω_n is the *magnitude of the poles*)

Settling Time in Frequency Domain

Suppose we want $t_s \leq c$

$$t_s \approx \frac{3}{\sigma} \leq c \qquad \Longrightarrow \qquad \sigma \geq \frac{3}{c}$$

Want poles to be sufficiently fast (large enough magnitude of real part):



Intuition: poles far to the left → transients decay faster → smaller t_s