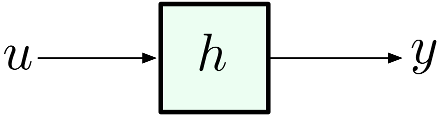
ECE486: Control Systems

- Lecture 3B: Calculating dynamic response with arbitrary I.C.'s Using Method of Partial Fractions

Goal: develop a methodology for characterizing the output of a given system for given input and initial conditions.

Dynamic Response



Problem: compute the response y to a given input u under a given set of initial conditions.

Both the input and the initial conditions can be arbitrary.

Laplace Transforms Revisited

Convolution: $\mathscr{L}\{f \star g\} = \mathscr{L}\{f\}\mathscr{L}\{g\}$
(useful because $Y(s) = H(s)U(s)$)

Example: $\dot{y} = -y + u \qquad y(0) = 0$

Compute the response for $u(t) = \cos t$

We already know

$$H(s) = \frac{1}{s+1}$$
$$U(s) = \frac{s}{s^2+1}$$

$$\Longrightarrow Y(s) = H(s)U(s) = \frac{s}{(s+1)(s^2+1)}$$
$$y(t) = \mathscr{L}^{-1}\{Y\}$$

— can't find $Y(s)$ in the tables. So how do we compute y?

Method of Partial Fractions

Problem: compute $\mathscr{L}^{-1}\left\{\frac{s}{(s+1)(s^2+1)}\right\}$

This Laplace transform is not in the tables, but let's look at the table anyway. What do we find?

$$\frac{1}{s+1} \qquad \mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t} \qquad (\#7)$$

$$\frac{1}{s^2+1} \qquad \mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t \qquad (\#17)$$

$$\frac{s}{s^2+1} \qquad \mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t \qquad (\#18)$$

— so we see some things that are similar to $Y(s)$, but not quite.

This brings us to the method of partial fractions:

- boring (i.e., character-building), but *very useful*
- allows us to break up complicated fractions into sums of simpler ones, for which we know $\mathscr{L}^{-1}$ from tables

Method of Partial Fractions

Problem: compute $\mathscr{L}^{-1}\{Y(s)\}$, where

$$Y(s) = \frac{s}{(s+1)(s^2+1)}$$

We seek a, b, c, such that

$$Y(s) = \frac{a}{s+1} + \frac{bs+c}{s^2+1} \quad \text{(need } bs+c \text{ so that deg(num) = deg(den)} - 1)$$

- Find a: multiply by $s+1$ to isolate a

$$(s+1)Y(s) = \frac{s}{s^2+1} = a + \frac{(s+1)(as+b)}{(s^2+1)}$$

— now let $s = -1$ to "kill" the second term on the RHS:

$$a = (s+1)Y(s)\Big|_{s=-1} = -\frac{1}{2}$$

Method of Partial Fractions

Problem: compute $\mathscr{L}^{-1}\{Y(s)\}$, where

$$Y(s) = \frac{s}{(s+1)(s^2+1)}$$

We seek a, b, c, such that

$$Y(s) = \frac{a}{s+1} + \frac{bs+c}{s^2+1} \quad \text{(need } bs+c \text{ so that deg(num) = deg(den)} - 1\text{)}$$

- Find b: multiply by s^2+1 to isolate $bs+c$

$$(s^2+1)Y(s) = \frac{s}{s+1} = \frac{a(s^2+1)}{s+1} + bs + c$$

— now let $s = j$ to "kill" the first term on the RHS:

$$bj + c = (s^2+1)Y(s)\Big|_{s=j} = \frac{j}{1+j}$$

Match Re(·) and Im(·) parts:

$$c + bj = \frac{j}{1+j} = \frac{j(1-j)}{(1+j)(1-j)} = \frac{1}{2} + \frac{j}{2} \quad \Longrightarrow b = c = \tfrac{1}{2}$$

Method of Partial Fractions

Problem: compute $\mathscr{L}^{-1}\{Y(s)\}$, where

$$Y(s) = \frac{s}{(s+1)(s^2+1)}$$

We found that

$$Y(s) = -\frac{1}{2(s+1)} + \frac{s}{2(s^2+1)} + \frac{1}{2(s^2+1)}$$

Now we can use linearity and tables:

$$\begin{aligned}
y(t) &= \mathscr{L}^{-1}\left\{-\frac{1}{2(s+1)} + \frac{s}{2(s^2+1)} + \frac{1}{2(s^2+1)}\right\} \\
&= -\frac{1}{2}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \\
&= -\frac{1}{2}e^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t \quad \text{(from tables)} \\
&= -\frac{1}{2}e^{-t} + \frac{1}{\sqrt{2}}\cos(t - \pi/4) \quad (\cos(a-b) = \cos a\cos b + \sin a\sin b)
\end{aligned}$$

Laplace Transforms and Differentiation

Given a differentiable function f, what is the Laplace transform $\mathscr{L}\{f'(t)\}$ of its time derivative?

$$\begin{aligned}\mathscr{L}\{f'(t)\} &= \int_0^\infty f'(t)e^{-st}\mathrm{d}t \\ &= f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)\mathrm{d}t \qquad \text{(integrate by parts)} \\ &= -f(0) + sF(s)\end{aligned}$$

— provided $f(t)e^{-st} \to 0$ as $t \to \infty$

$\mathscr{L}\{f'(t)\} = sF(s) - f(0)$ — this is how we account for I.C.'s

Similarly:

$$\begin{aligned}\mathscr{L}\{f''(t)\} = \mathscr{L}\{(f'(t))'\} &= s\mathscr{L}\{f'(t)\} - f'(0) \\ &= s^2F(s) - sf(0) - f'(0)\end{aligned}$$

Example

Consider the system

$$\ddot{y} + 3\dot{y} + 2y = u, \qquad y(0) = \dot{y}(0) = 0$$

(need two I.C.'s for 2nd-order ODE's)

Let's compute the transfer function: $H(s) = \dfrac{Y(s)}{U(s)}$

— take Laplace transform of both sides (zero I.C.'s):

$$s^2 Y(s) + 3sY(s) + 2Y(s) = U(s) \qquad H(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + 3s + 2}$$

Example (continued)

$$\ddot{y} + 3\dot{y} + 2y = u, \qquad y(0) = \alpha,\ \dot{y}(0) = \beta$$

Compute the *step response*, i.e., response to $u(t) = 1(t)$

Caution!! $Y(s) = H(s)U(s)$ no longer holds if $\alpha \neq 0$ or $\beta \neq 0$

Again, take Laplace transforms of both sides, mind the I.C.'s:

$$s^2 Y(s) - s\alpha - \beta + 3sY(s) - 3\alpha + 2Y(s) = U(s)$$

$U(s) = \mathscr{L}\{1(t)\} = 1/s$, which gives

$$s^2 Y(s) - s\alpha - \beta + 3sY(s) - 3\alpha + 2Y(s) = \frac{1}{s}$$

$$Y(s) = \frac{\alpha s + (3\alpha + \beta) + \frac{1}{s}}{s^2 + 3s + 2} = \frac{\alpha s^2 + (3\alpha + \beta)s + 1}{s(s+1)(s+2)}$$

Note: if $\alpha = \beta = 0$, then $Y(s) = \dfrac{1}{s(s+1)(s+2)} = H(s)U(s)$

Example (continued)

Compute the step response of

$$\ddot{y} + 3\dot{y} + 2y = u, \qquad y(0) = \alpha,\ \dot{y}(0) = \beta$$

$$Y(s) = \frac{\alpha s^2 + (3\alpha + \beta)s + 1}{s(s+1)(s+2)} \qquad y(t) = \mathscr{L}^{-1}\{Y(s)\}$$

Use the method of partial fractions:

$$\frac{\alpha s^2 + (3\alpha + \beta)s + 1}{s(s+1)(s+2)} = \frac{a}{s} + \frac{b}{s+1} + \frac{c}{s+2}$$

— this gives $a = 1/2,\ b = 2\alpha + \beta - 1,\ c = -\alpha - \beta + 1/2$

$$Y(s) = \frac{1}{2s} + (2\alpha + \beta - 1)\frac{1}{s+1} + \frac{-\alpha - \beta + 1/2}{s+2}$$

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = \frac{1}{2}1(t) + (2\alpha + \beta - 1)e^{-t} + (1/2 - \alpha - \beta)e^{-2t}$$

Example (continued)

The step response of

$$\ddot{y} + 3\dot{y} + 2y = u, \qquad y(0) = \alpha,\ \dot{y}(0) = \beta$$

is given by

$$y(t) = \frac{1}{2}1(t) + (2\alpha + \beta - 1)e^{-t} + (1/2 - \alpha - \beta)e^{-2t}$$

What are the transient and the steady-state terms?

- The transient terms are e^{-t}, e^{-2t} (decay to zero at exponential rates -1 and -2)

  Note the poles of $H(s) = \dfrac{1}{(s+1)(s+2)}$ at $s = -1$ and $s = -2$

  — these are *stable poles* (both lie in LHP)
- the steady-state part is $\frac{1}{2}1(t)$ — converges to steady-state value of $1/2$