ECE 486: Control Systems

Lecture 25B: Linearization

Key Takeaways

This lecture presents a method to approximate a nonlinear state-space model by a linear state-space model.

This process is known as Jacobian linearization and involves:

1. Compute an equilibrium point. This is essentially a constant solution to the nonlinear system. This is known as trimming the system.
2. Approximate the nonlinear dynamics near the equilibrium point using a Taylor series expansion.

A nonlinear system can have many equilibrium points and each one can have a different linear state-space approximation.

Taylor Series Approximation

Jacobian linearization relies on a Taylor series expansion.

Consider a scalar function $f: \boldsymbol{R} \rightarrow \boldsymbol{R}$.

The Taylor series of f at a point $\bar{x} \in \boldsymbol{R}$ is:

$$f(x) = f(\bar{x}) + \frac{df}{dx}(\bar{x}) \cdot (x - \bar{x}) + \text{Higher Order Terms (Quadratic, etc)}$$

If x is near $\bar{x}$ then the higher order terms can be neglected. This yields a linear function that approximates f:

$$f(x) \approx f(\bar{x}) + \frac{df}{dx}(\bar{x}) \cdot (x - \bar{x})$$

Line with slope $\frac{df}{dx}(\bar{x})$ passing through $(\bar{x}, f(\bar{x}))$.

The error in making this linear approximation is on the order of $(x - \bar{x})^2$. This error is “small“ if x is near $\bar{x}$.

Example

The wind drag on a car is given by:

$$f(v) = c_D v^2 \text{ where } v \text{ is the velocity (m/s) and } c_D = 0.4\frac{Ns^2}{m^2}$$

The Taylor series approximation near $\bar{v} = 29\frac{m}{s}$ is:

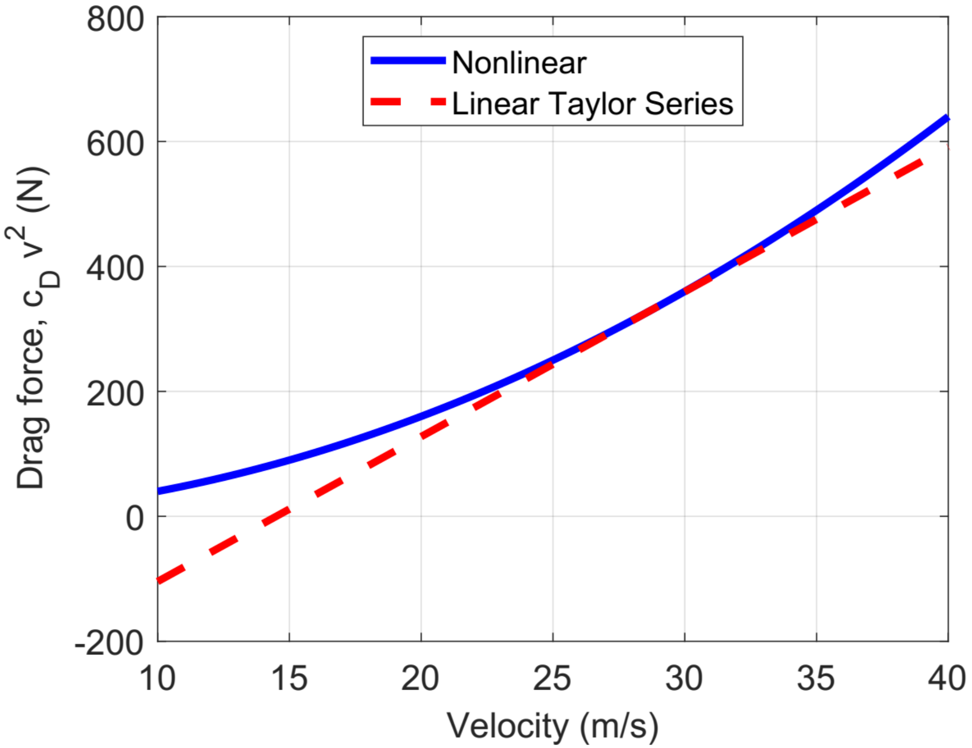
$$f(v) \approx c_D\bar{v}^2 + (2c_D\bar{v})\cdot(v-\bar{v}) \;= 336.4N + \left(23.2\frac{N\cdot sec}{m}\right)\cdot\left(v - 29\frac{m}{sec}\right)$$

At 30m/s:

- *f(30)* = 360N
- Linear Taylor Series = 359.6N

At 10m/s:

- *f(10)*=40
- Linear Taylor Series = -104.4N



Multivariable Taylor Series

Consider a multivariable function $f: \boldsymbol{R}^n \rightarrow \boldsymbol{R}^m$.

Let $\frac{df}{dx}$ denote the m-by-n matrix whose (i,j) entry is $\frac{\partial f_i}{\partial x_j}$.

This is called the Jacobian matrix.

The Taylor series of f at a point $\bar{x} \in \boldsymbol{R}^n$ is:

$$f(x) = f(\bar{x}) + \frac{df}{dx}(\bar{x}) \cdot (x - \bar{x}) + \text{Higher Order Terms (Quadratic, etc)}$$

If x is near $\bar{x}$ then the higher order terms can be neglected.
This yields a linear function that approximates f:

$$f(x) \approx f(\bar{x}) + \frac{df}{dx}(\bar{x}) \cdot (x - \bar{x})$$

Example

Consider a multivariable function $f: \boldsymbol{R}^3 \rightarrow \boldsymbol{R}^2$:

$$f(x) = \begin{bmatrix} 3x_1^2 - \sin(x_2) \\ x_3 + 5x_1x_2 - 9x_1x_3^2 \end{bmatrix}$$

The 2-by-3 Jacobian matrix of partial derivatives is:

$$\frac{df}{dx}(x) = \begin{bmatrix} 6x_1 & -\cos(x_2) & 0 \\ 5x_2 - 9x_3^2 & 5x_1 & 1 - 18x_1x_3 \end{bmatrix}$$

The Taylor series of f at $\bar{x} = [-2, 0, 7]^T$ is:

$$f(x) \approx \begin{bmatrix} 12 \\ 889 \end{bmatrix} + \begin{bmatrix} -12 & -1 & 0 \\ -441 & -10 & 253 \end{bmatrix} \cdot \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \\ 7 \end{bmatrix} \right)$$

If $x = [-2.1, 0.2, 6.8]^T$ then

- $f(x) = [13.03, 878.64]^T$
- Linear Taylor Series = $[13.00, 880.5]^T$

The approximation is accurate if $x \approx \bar{x}$

Equilibrium (Trim) Points

Consider an nth-order, nonlinear, SISO state-space model:

$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t), u(t))\end{aligned} \qquad \text{IC: } x(0) = x_0$$

An **equilibrium point** consists of constant values $\bar{x} \in \boldsymbol{R}^n$, $\bar{u} \in \boldsymbol{R}$, and $\bar{y} \in \boldsymbol{R}$ such that:

$$\begin{aligned}0 &= f(\bar{x}, \bar{u}) \\ \bar{y} &= h(\bar{x}, \bar{u})\end{aligned}$$

If $u(t) = \bar{u}$ for $t \geq 0$ and $x(0) = \bar{x}$ then $x(t) = \bar{x}$ and $y(t) = \bar{y}$ solves the nonlinear state-space model.

Finding an equilibrium is called "trimming" the system. There are *n+1* equations and *n+2* unknowns. Hence the equilibrium point is typically not unique. (Matlab function: `trim`)

Example

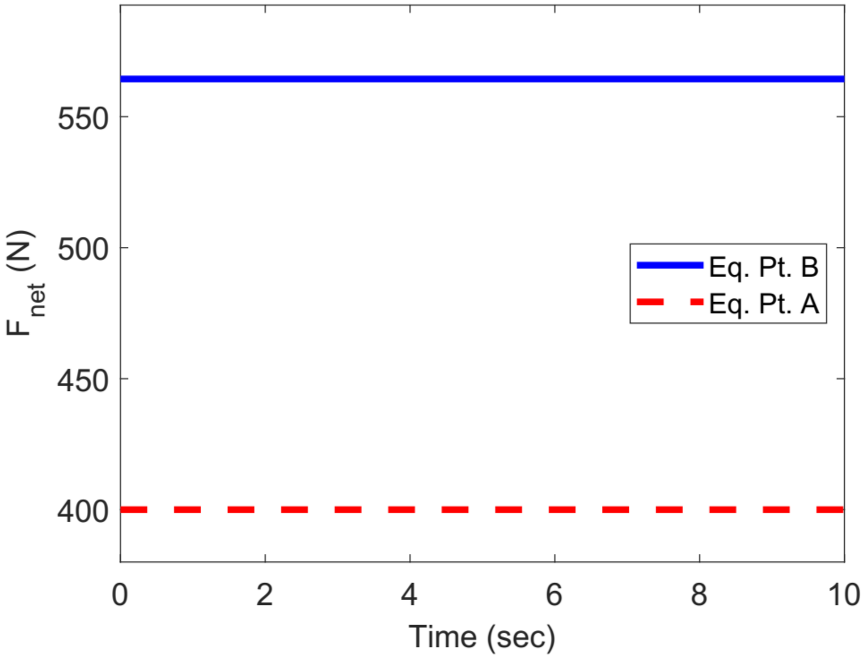
Consider the following model for a car:

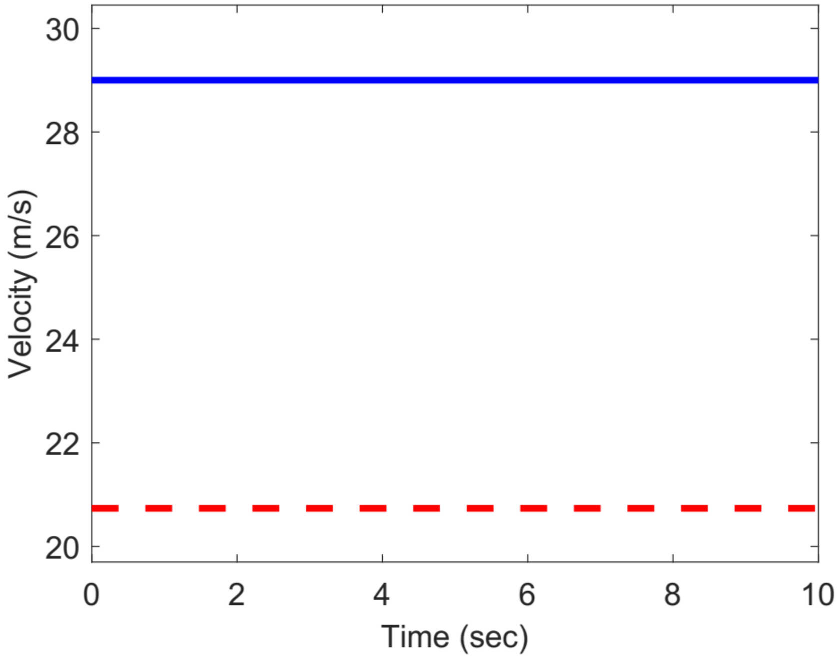
$$\dot{v}(t) = \frac{1}{2085}\left(F_{net}(t) - 0.4v^2(t) - 228\right) := f(v(t), F_{net}(t)) \qquad \text{IC: } v(0) = v_0$$

$$y(t) = v(t)$$

An equilibrium point $(\bar{v}, \bar{F}_{net})$ satisfies $f(\bar{v}, \bar{F}_{net}) = 0$ and $\bar{y} = \bar{v}$.

- Equilibrium Point A: $(\bar{v}, \bar{F}_{net}) = (20.7, 400)$
- Equilibrium Point B: $(\bar{v}, \bar{F}_{net}) = (29, 564)$





Jacobian Linearization

Consider an n^th^-order, nonlinear, SISO state-space model:

$$\dot{x}(t) = f(x(t), u(t)) \qquad \text{IC: } x(0) = x_0$$
$$y(t) = h(x(t), u(t))$$

Assume $(\bar{x}, \bar{u}, \bar{y})$ is an equilibrium point: $f(\bar{x}, \bar{u}) = 0, \bar{y} = h(\bar{x}, \bar{u})$.

Jacobian linearization is used to approximate the solution $(x(t), y(t))$ to the nonlinear state-space model when $x(0)$ is slightly different from $\bar{x}$ and/or the input u(t) is slightly different from $\bar{u}$. (Matlab function: `linearize`)

Jacobian Linearization

Consider an nth-order, nonlinear, SISO state-space model:

$$\dot{x}(t) = f(x(t), u(t)) \qquad \text{IC: } x(0) = x_0$$
$$y(t) = h(x(t), u(t))$$

Assume $(\bar{x}, \bar{u}, \bar{y})$ is an equilibrium point: $f(\bar{x}, \bar{u}) = 0, \bar{y} = h(\bar{x}, \bar{u})$.

1. Define deviations from the equilibrium point:

$$\delta_x(t) := x(t) - \bar{x}, \qquad \delta_u(t) := u(t) - \bar{u}, \qquad \delta_y(t) := y(t) - \bar{y}$$

2. Re-write nonlinear state-space model using deviations:

$$\dot{\delta}_x(t) = \dot{x}(t) \Rightarrow \quad \begin{aligned} \dot{\delta}_x(t) &= f(\bar{x} + \delta_x(t), \bar{u} + \delta_u(t)) \\ \delta_y(t) &= h(\bar{x} + \delta_x(t), \bar{u} + \delta_u(t)) - \bar{y} \end{aligned}$$

3. Use linear Taylor series approximation:

$$f(\bar{x} + \delta_x(t), \bar{u} + \delta_u(t)) \approx f(\bar{x}, \bar{u}) + \frac{\partial f}{\partial x}(\bar{x}, \bar{u}) \cdot \delta_x + \frac{\partial f}{\partial u}(\bar{x}, \bar{u}) \cdot \delta_u \quad = A\delta_x(t) + B\delta_u(t)$$

$$h(\bar{x} + \delta_x(t), \bar{u} + \delta_u(t)) \approx h(\bar{x}, \bar{u}) + \frac{\partial h}{\partial x}(\bar{x}, \bar{u}) \cdot \delta_x + \frac{\partial h}{\partial u}(\bar{x}, \bar{u}) \cdot \delta_u \quad = \bar{y} + C\delta_x(t) + D\delta_u(t)$$

Jacobian Linearization

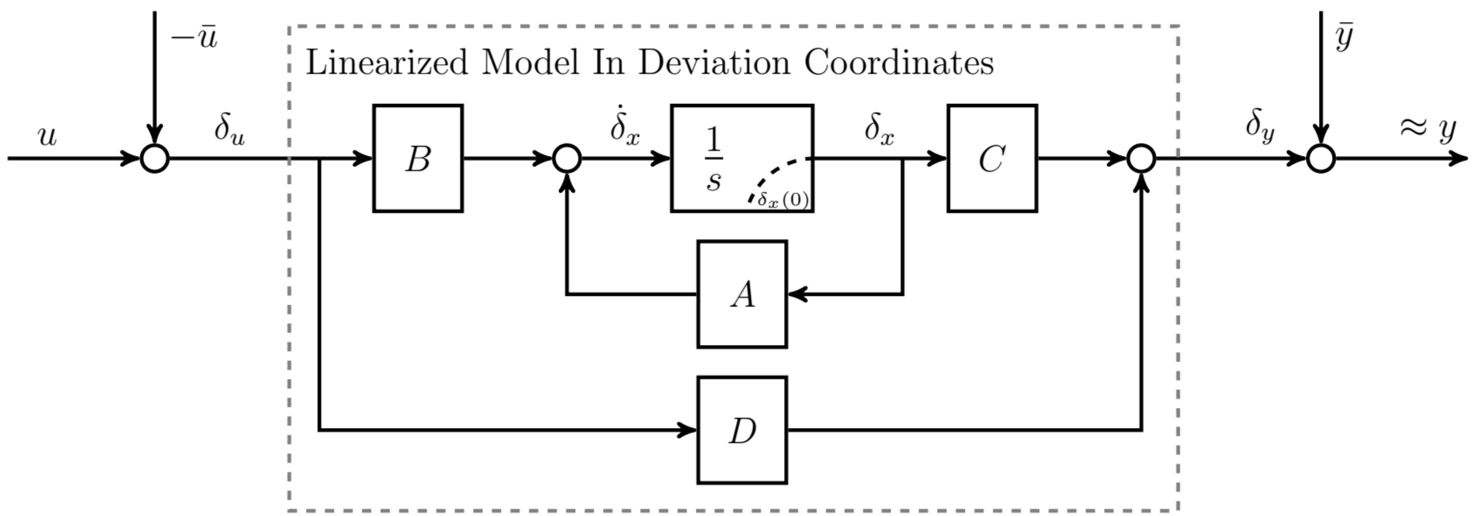
Consider an n^th-order, nonlinear, SISO state-space model:

$$\dot{x}(t) = f(x(t), u(t)) \qquad \text{IC: } x(0) = x_0$$
$$y(t) = h(x(t), u(t))$$

Assume $(\bar{x}, \bar{u}, \bar{y})$ is an equilibrium point: $f(\bar{x}, \bar{u}) = 0, \bar{y} = h(\bar{x}, \bar{u})$.

Linear state-space approximation is:

$$\dot{\delta}_x(t) = A\delta_x(t) + B\delta_u(t) \qquad \text{IC: } \delta_x(0) = x_0 - \bar{x}$$
$$\delta_y(t) = C\delta_x(t) + D\delta_u(t)$$



Validating the Linearization

To verify that you have correctly constructed the linear state-space approximation:

1. Let $(x(t), y(t))$ be a solution of the nonlinear system with:
   - Initial condition $x(0) = x_0$ near $\bar{x}$
   - Input $u(t)$ that remains near $\bar{u}$.
2. Let $(\delta_x(t), \delta_y(t))$ be a solution of the linear approximation with:
   - Initial condition $\delta_x(0) = x_0 - \bar{x}$
   - Input $\delta_u(t) = u(t) - \bar{u}$
3. Shift linear solution from deviation coordinates $(\delta_x(t), \delta_y(t))$ back to original coordinates:

$$\left(x_{Lin}(t), y_{Lin}(t)\right) = (\delta_x(t) + \bar{x}, \delta_y(t) + \bar{y})$$

The solutions $(\boldsymbol{x}(\boldsymbol{t}), \boldsymbol{y}(\boldsymbol{t}))$ and $\left(\boldsymbol{x}_{\boldsymbol{Lin}}(\boldsymbol{t}), \boldsymbol{y}_{\boldsymbol{Lin}}(\boldsymbol{t})\right)$ should be close (at least over a short time-horizon).

Example

Consider the following model for a car:

$$\dot{v}(t) = \frac{1}{2085}\left(F_{net}(t) - 0.4v^2(t) - 228\right) := f(v(t), F_{net}(t)) \qquad \text{IC: } v(0) = v_0$$

$$y(t) = v(t)$$

with equilibrium point $(\bar{v}, \bar{F}_{net}) = (20.7{,}400)$.

Partial derivatives are:

$$\frac{\partial f}{\partial v}(\bar{v}, \bar{F}_{net}) = \left.\frac{-2 \cdot 0.4\bar{v}}{2085}\right|_{(\bar{v}, \bar{F}_{net})} = -0.008\frac{1}{sec}$$

$$\frac{\partial f}{\partial F_{net}}(\bar{v}, \bar{F}_{net}) = \left.\frac{1}{2085}\right|_{(\bar{v}, \bar{F}_{net})} = 4.8 \times 10^{-4}\frac{1}{kg}$$

Linearization is:

$$\dot{\delta}_v(t) = -0.008\delta_v(t) + \left(4.8 \times 10^{-4}\right)\delta_F(t)$$

$$\delta_y(t) = \delta_v(t)$$

$$G(s) = \frac{4.8 \times 10^{-4}}{s + 0.008}.$$

Example

Consider the following model for a car:

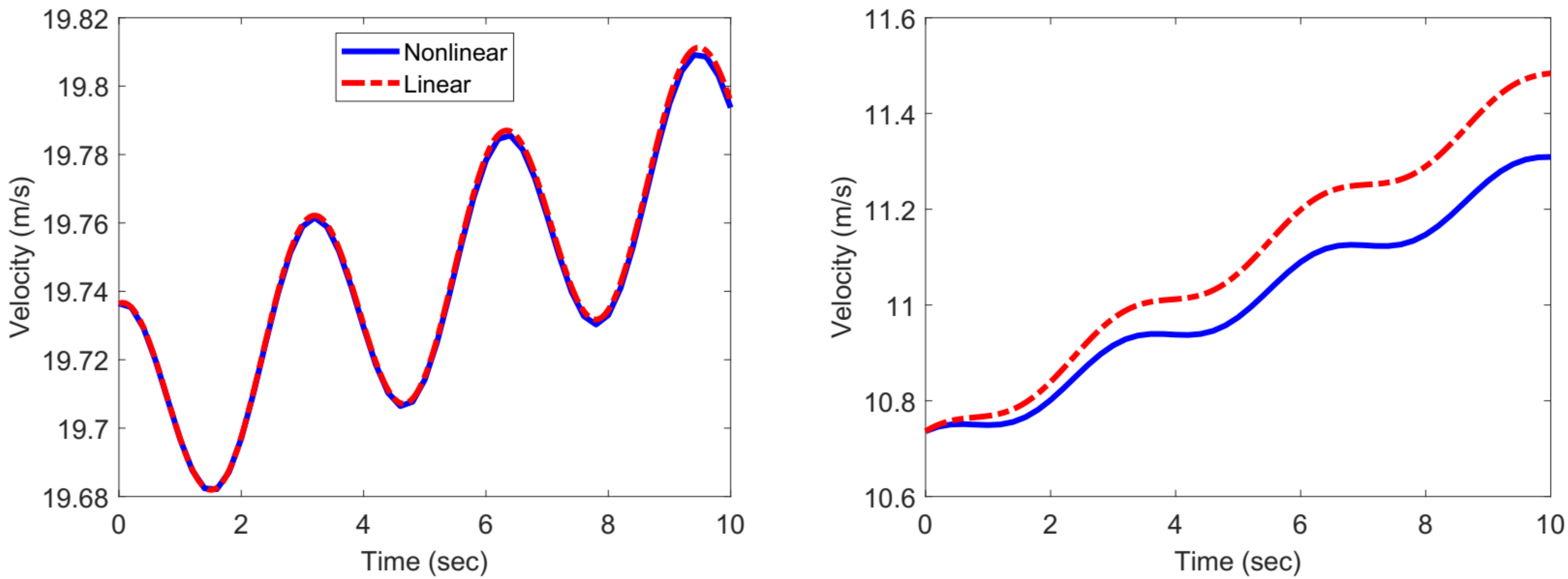
$$\dot{v}(t) = \frac{1}{2085}\left(F_{net}(t) - 0.4v^2(t) - 228\right) := f(v(t), F_{net}(t)) \qquad \text{IC: } v(0) = v_0$$

$$y(t) = v(t)$$

with equilibrium point $(\bar{v}, \bar{F}_{net}) = (20.7{,}400)$.

Simulation Input: $F_{net}(t)=400-140\sin(2t)$ (N)

Initial Conditions: v(0)=19.7m/s (left) and v(0)=10.74m/s (right)



Jacobian Linearization

- If the solution of a nonlinear state-space model remains “near" an equilibrium point then the dynamics can be approximated by a linear state-space model.

- The linear state-space model can then be converted to a linear ODE / transfer function representation.

- This is useful because:
  - Standard control design methods make use of linear ODE models and their corresponding transfer functions.
  - Many controllers are designed to keep a system near a particular equilibrium point. Hence, the controller will ensure the linear approximation is valid.