ECE 486: Control Systems

Lecture 16A: Sensitivity Functions

Key Takeaways

This lectures considers a generic feedback system with plant *G(s)* and controller *K(s)*.

Two important transfer functions are:

- Sensitivity: $S(s) = \frac{1}{1+G(s)K(s)}$
- Complementary Sensitivity: $T(s) = \frac{G(s)K(s)}{1+G(s)K(s)}$

The feedback system is defined to be stable if every possible input/output transfer function is internally stable.

- This holds if and only if all zeros of 1+G(s)K(s) are in the LHP.
- The feedback system is unstable if the G(s)K(s) has a pole/zero cancellation in the CRHP.

Generic Feedback System

The feedback system below has:

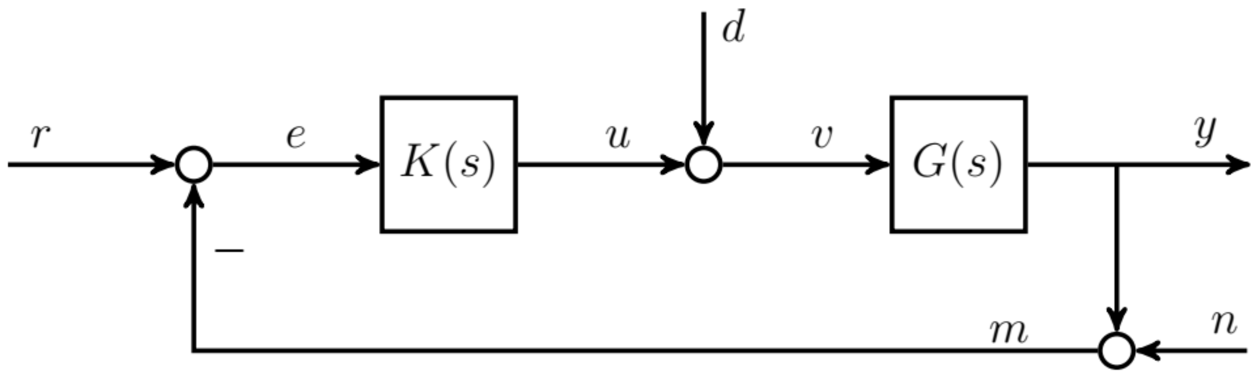
- Inputs: reference *r*, disturbance *d*, and sensor noise *n*
- "Internal" Signals: error *e*, control command *u*, plant input *v*, plant output *y*, measurement *m*.

There are many possible input/output pairs. Two examples:

$$T_{r\to y}(s) = \frac{G(s)K(s)}{1+G(s)K(s)} \quad \text{and} \quad T_{r\to e}(s) = \frac{1}{1+G(s)K(s)}$$

Complementary Sensitivity, *T(s)* Sensitivity, *S(s)*

S and *T* are complementary in the sense that *T(s)+S(s)=1* for all *s*.



Generic Feedback System

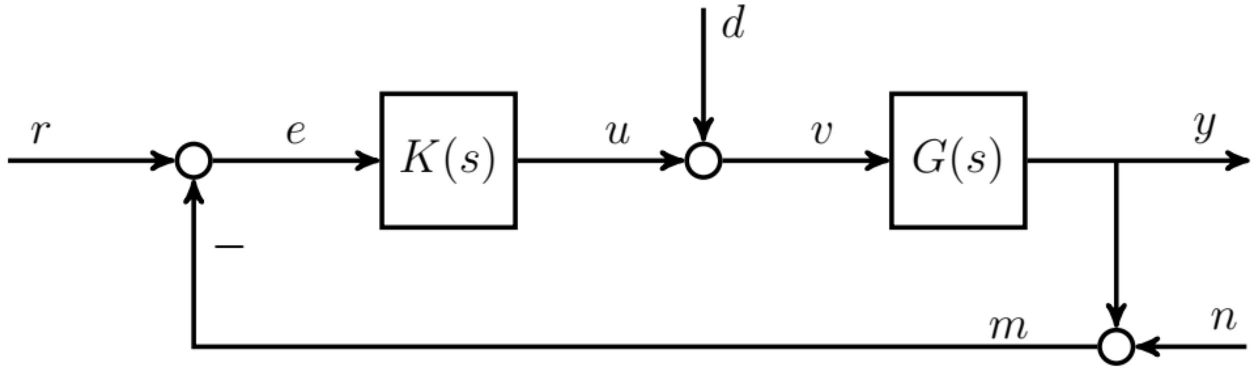
All possible transfer functions from the three inputs to the five internal signals are given by:

$$\begin{bmatrix} Y(s) \\ U(s) \\ E(s) \\ M(s) \\ V(s) \end{bmatrix} = \begin{bmatrix} T(s) & -T(s) & G(s)S(s) \\ K(s)S(s) & -K(s)S(s) & -T(s) \\ S(s) & -S(s) & -G(s)S(s) \\ T(s) & S(s) & G(s)S(s) \\ K(s)S(s) & -K(s)S(s) & S(s) \end{bmatrix} \begin{bmatrix} R(s) \\ N(s) \\ D(s) \end{bmatrix}$$

The array is interpreted using matrix/vector multiplication:

$$U(s) = K(s)S(s)R(s) - K(s)S(s)N(s) - T(s)D(s)$$

The effects of each input sum together by linear superposition.



Closed-Loop Stability

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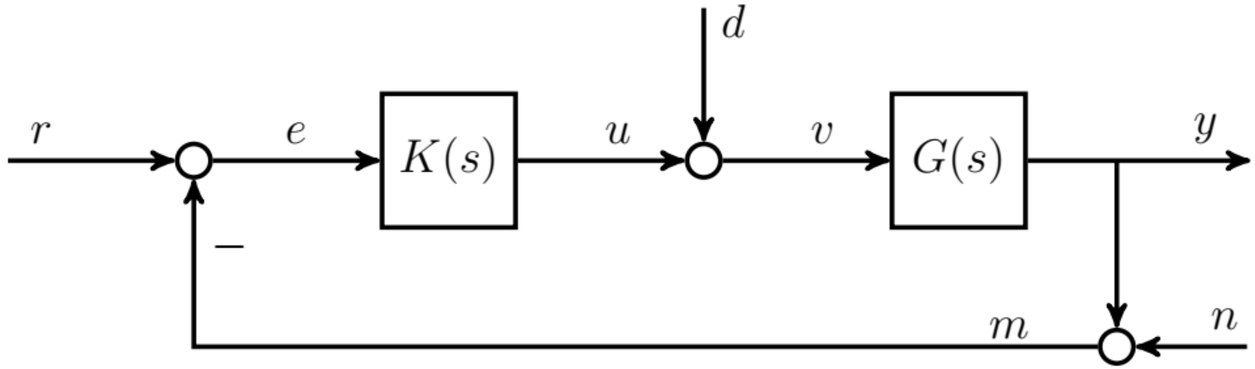
Definition: The feedback system is **stable** if all transfer functions in the system (*r* to *e*, *d* to *u*, *n* to *y*, etc) are internally stable.

Stability of the feedback system is distinct from stability of G(s) and/or K(s).



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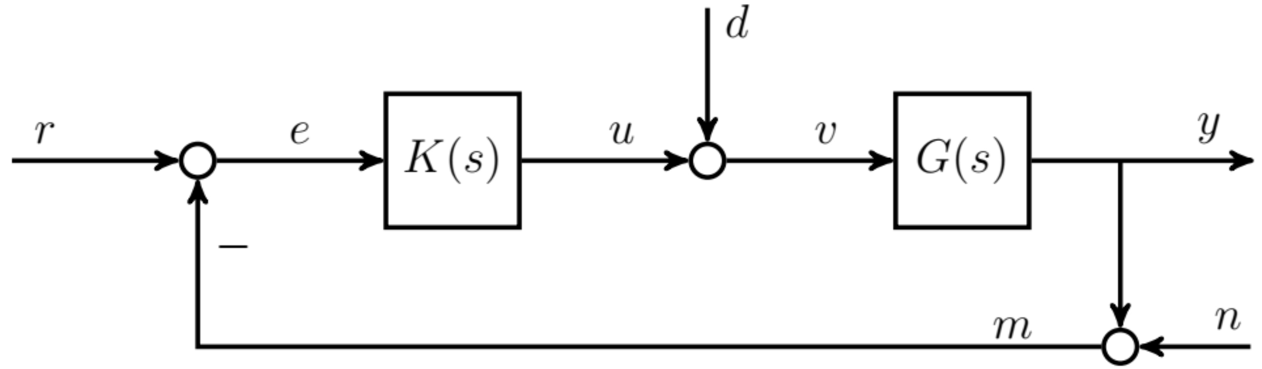
The 5-by-3 array contains only four unique entries:

$$\pm S(s), \pm T(s), \pm K(s)S(s), \text{ and } \pm G(s)S(s)$$



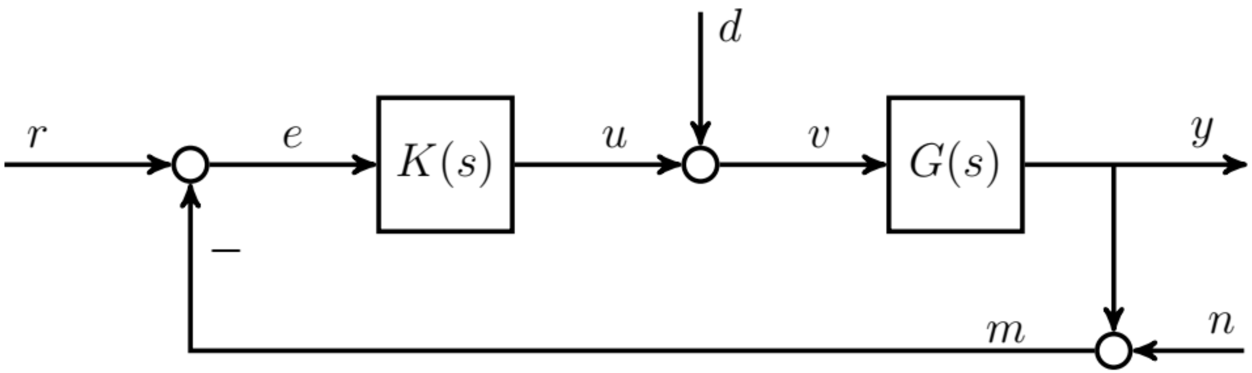
Condition for Closed-Loop Stability

The feedback system is stable if and only if all of the following transfer functions are stable:

$$S(s) = \frac{1}{1+G(s)K(s)}, \qquad T(s) = \frac{G(s)K(s)}{1+G(s)K(s)},$$

$$G(s)S(s) = \frac{G(s)}{1+G(s)K(s)}, \qquad K(s)S(s) = \frac{K(s)}{1+G(s)K(s)}.$$

Fact: The closed-loop is stable if and only if all zeros of 1+G(s)K(s) are in the LHP.



Pole/Zero Cancellations in the CRHP

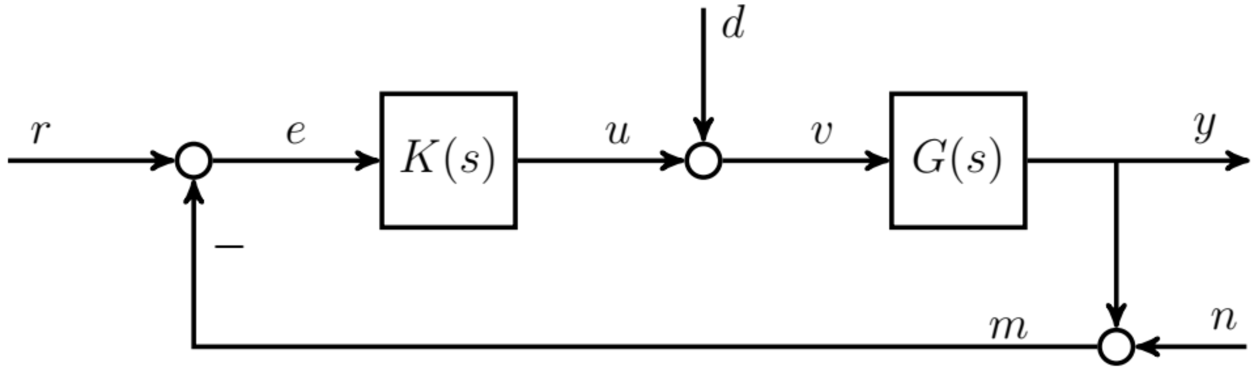
Consider the feedback system with:

$$K(s) = \frac{1}{s-1} \text{ and } G(s) = \frac{s-1}{s+3}$$

The closed-loop poles are given by the zeros of:

$$1 + G(s)K(s) = \frac{(s+3)\cdot(s-1) + (s-1)\cdot 1}{(s-1)(s+3)} = \frac{(s-1)(s+4)}{(s-1)(s+3)}$$

This has a zero at s=+1 and hence the closed-loop is unstable.



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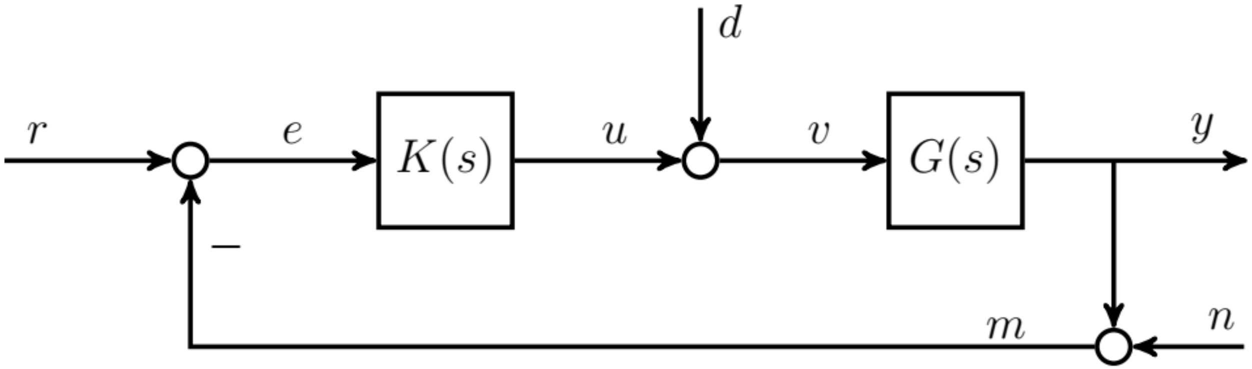
Consider the feedback system with:

$$K(s) = \frac{1}{s-1} \text{ and } G(s) = \frac{s-1}{s+3}$$

The pole/zero cancellation in the product *G(s)K(s)* hides the instability in some input/output transfer functions:

$$T_{r\to y}(s) = \frac{G(s)K(s)}{1+G(s)K(s)} = \frac{s-1}{(s-1)(s+4)} \qquad T_{r\to u}(s) = \frac{K(s)}{1+G(s)K(s)} = \frac{s+3}{(s-1)(s+4)}$$



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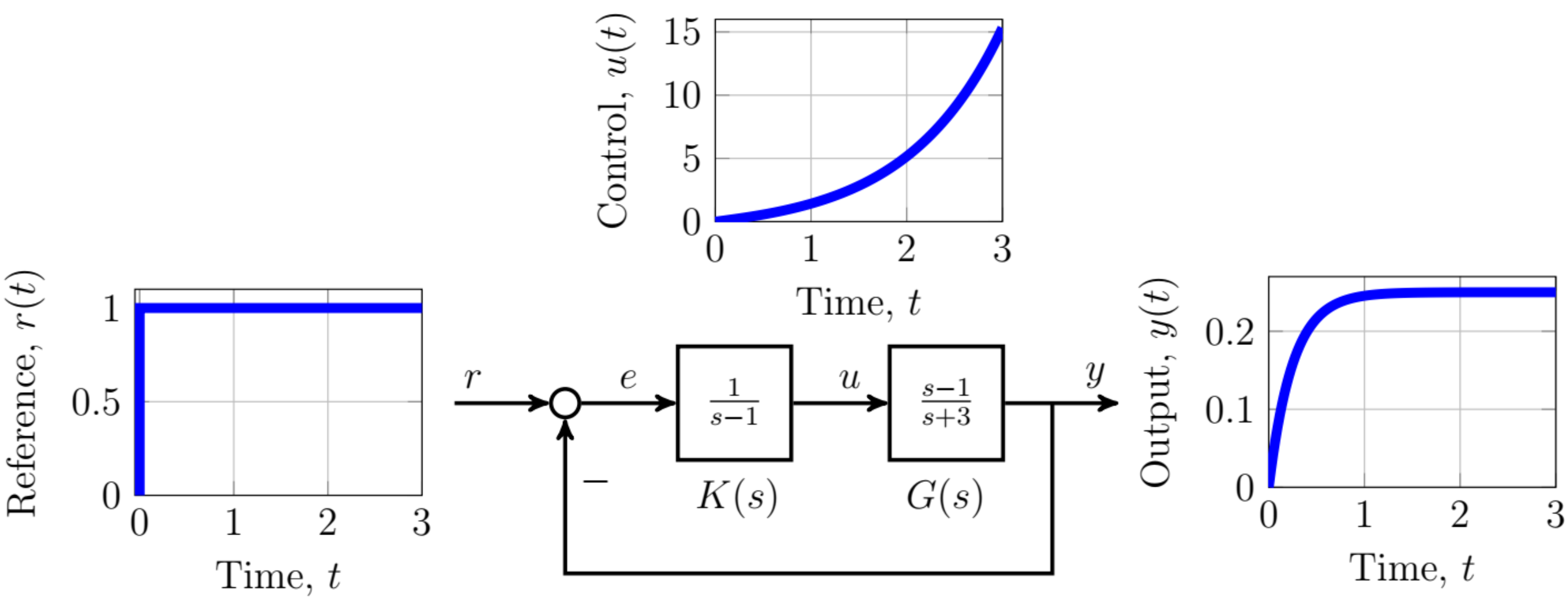

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Never design a controller to cancel a CRHP pole or zero in the plant. This will always result in an unstable feedback system.