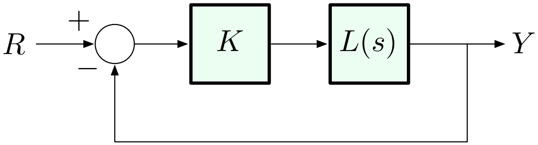
ECE486: Control Systems

- Lecture 12B: Case Study on Control Design

Goal: learn how to use Root Locus to select control gains.

Reading: FPE, Chapter 5

Reminder: Root Locus



where $L(s) = \dfrac{b(s)}{a(s)} = \dfrac{s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n}$, $m \leq n$

Root locus: the set of all $s \in \mathbb{C}$ that solve the *characteristic equation*

$$a(s) + Kb(s) = 0$$

as K varies from 0 to ∞.

Using RL to Select Parameter Values

In Lab 5, you will need to select the value of gain K that corresponds to a desired pole on the root locus.

Here is one way of doing it:

$$L(s) = -\frac{1}{K} \qquad \text{— negative real number}$$

$$\Updownarrow$$

$$K = -\frac{1}{L(s)} = \frac{1}{|L(s)|}$$

$$L(s) = \frac{(s - z_1)\dots(s - z_m)}{(s - p_1)\dots(s - p_n)}$$

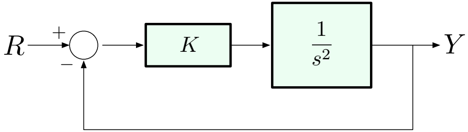
$$\Longrightarrow \quad K = \frac{1}{|L(s)|} = \frac{|s - p_1|\dots|s - p_n|}{|s - z_1|\dots|s - z_m|}$$

Control Design Using Root Locus

Case study: double integrator, transfer function $G(s) = \dfrac{1}{s^2}$

Control objective: ensure stability; meet time response specs.

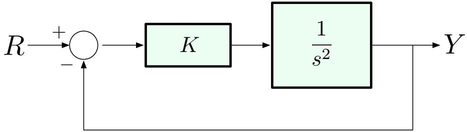
First, let's try a simple P-gain:



Closed-loop transfer function:

$$\frac{\frac{K}{s^2}}{1 + \frac{K}{s^2}} = \frac{K}{s^2 + K}$$

Double Integrator with P-Gain



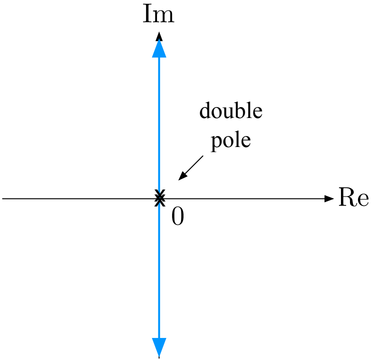
Closed-loop transfer function:

$$\frac{\frac{K}{s^2}}{1+\frac{K}{s^2}} = \frac{K}{s^2+K}$$

Characteristic equation:

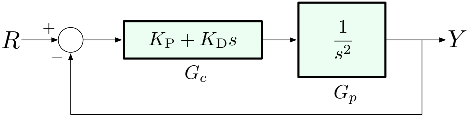
$$s^2 + K = 0$$

Closed-loop poles: $s = \pm\sqrt{K}j$



This confirms what we already knew: P-gain alone does not deliver stability.

Double Integrator with PD-Control



Characteristic equation: $$1 + \underbrace{(K_\mathrm{P} + K_\mathrm{D} s)}_{G_c(s)} \cdot \underbrace{\frac{1}{s^2}}_{G_p(s)} = 0$$

$$s^2 + K_\mathrm{D} s + K_\mathrm{P} = 0$$

To use the RL method, we need to convert it into the Evans form $1 + KL(s) = 0$, where $L(s) = \frac{b(s)}{a(s)} = \frac{s^m + b_1 s^{m-1} + \dots}{s^n + a_1 s^{n-1} + \dots}$

$$1 + (K_\mathrm{P} + K_\mathrm{D} s)\frac{1}{s^2} = 1 + K_\mathrm{D} \cdot \frac{s + K_\mathrm{P}/K_\mathrm{D}}{s^2}$$

$$\Longrightarrow \; K = K_\mathrm{D}, \; L(s) = \frac{s + K_\mathrm{P}/K_\mathrm{D}}{s^2} \qquad \text{(assume } K_\mathrm{P}/K_\mathrm{D} \text{ fixed, } = 1\text{)}$$

Double Integrator with PD-Control

Characteristic equation: $1 + K \cdot \frac{s+1}{s^2} = 0$

Here we can still write out the roots explicitly:

$$s^2 + Ks + K = 0 \qquad \Longrightarrow \qquad s = \frac{-K \pm \sqrt{K^2 - 4K}}{2}$$

But let's actually draw the RL using the rules:

Rule A: 2 branches

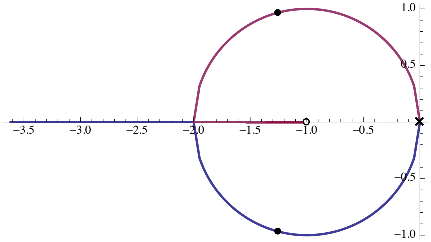
Rule B: both start at $s = 0$

Rule C: one ends at $z_1 = -1$, the other at ∞

Rule D: one branch will go off to $-\infty$

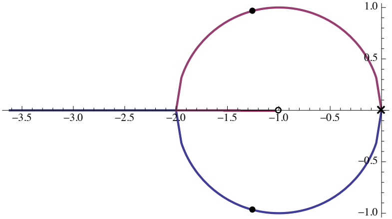
Rule E: asymptote angles at 180°

Rule F: no $j\omega$-crossings except for $s = p_1 = p_2 = 0$



Double Integrator with PD-Control

Characteristic equation: $$1 + K \cdot \frac{s+1}{s^2} = 0$$



What can we conclude from this root locus about stabilization?

- all closed-loop poles are in LHP (we already knew this from Routh, but now can visualize)
- nice damping, so can meet reasonable specs

So, the effect of D-gain was to introduce an *open-loop zero* into LHP, and this zero "pulled" the root locus into LHP, thus stabilizing the system.