

1. Consider the plant with transfer function $G(s) = \frac{1}{s}$ connected in standard feedback configuration with the controller $D(s) = k_1 + \frac{k_2}{s+1}$ where k_1 and k_2 are some parameters.

a) Rewrite the closed-loop characteristic equation in the form $1 + KL(s) = 0$, suitable for the root locus method. You can assume for simplicity that $k_1 = k_2$.

Solution:

The closed-loop characteristic equation is:

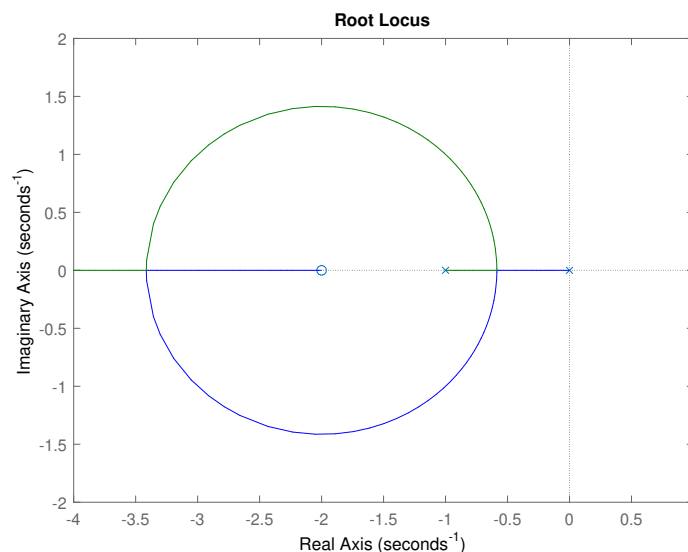
$$1 + D(s)G(s) = 0 \Rightarrow 1 + \left(k_1 + \frac{k_2}{s+1}\right) \frac{1}{s} = 0 \Rightarrow 1 + k_1 \frac{s+1 + \frac{k_2}{k_1}}{s(s+1)} = 0$$

For $K = k_1 = k_2$,

$$1 + K \frac{s+2}{s(s+1)} = 0 \Rightarrow L(s) = \frac{s+2}{s(s+1)}$$

b) Sketch the (positive) root locus for $L(s)$. Explain what rules you used to plot it. Describe the behavior of the root locus branches as K varies from 0 to $+\infty$.

Solution:



Rule 1: 2 branches start at poles, one goes to zero at -2 , the other goes to infinity;

Rule 2: the root locus located on real axis is shown;

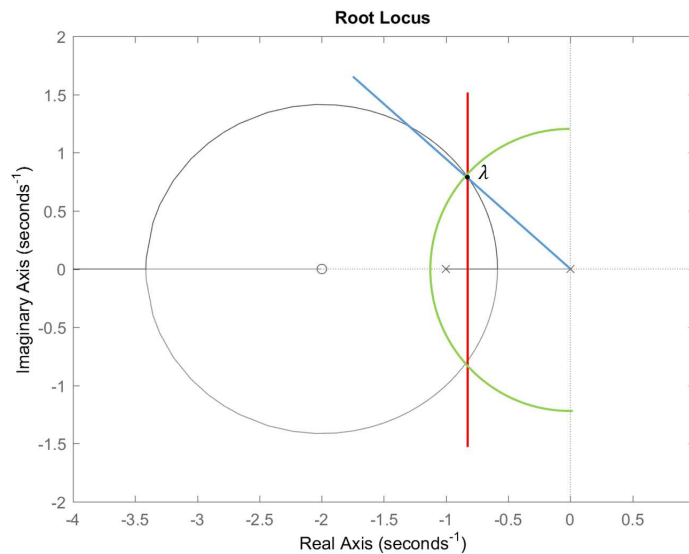
Rule 3 and 4: 1 asymptote which is on the negative real axis and it is drawn already;

Rule 5: $s^2 + (K + 1)s + 2 = 0$ has two roots on LHP for all $k > -1$, so there is no $j\omega$ -axis crossings;
 Rule 6: points of multiple roots: $s(s + 1) = (2s + 1)(s + 2) \Leftrightarrow s = -2 \pm \sqrt{2}$

c) Based on the root locus, discuss what values of the rise time, overshoot, and settling time are achievable by picking appropriate gain K . If exactly calculating the best values is difficult for some of these specs, it is enough if you explain graphically whether arbitrarily good values of the specs are achievable or not, and why.

Solution:

For a root λ on the root locus showing in the figure below, the corresponding rise time, overshoot and settling time are related to the radius of green arc, slope of blue line and distance between red line and $j\omega$ -axis, respectively.



Rise time: as $t_r \approx \frac{1.8}{\omega_n}$, minimal rise time is achieved when the roots are farthest away from origin. In this case, this is when both of the roots come together at $s = -2 - \sqrt{2}$. So best $t_r \approx \frac{1.8}{3.4} \approx 0.53$. In this case K is found to be $3 + 2\sqrt{2}$.

Overshoot: By picking K large enough or small enough such that both roots are on the negative real axis, the system is perfectly damped. So the minimal M_p is 0. On the other hand, largest M_p is achieved when the blue line is tangent to the root locus, in which case it is found that $\frac{\zeta}{\sqrt{1-\zeta^2}} = \tan 45^\circ = 1 \Rightarrow M_p = e^{-\pi} \approx 4.32\%$. In this case K is found to be 1.

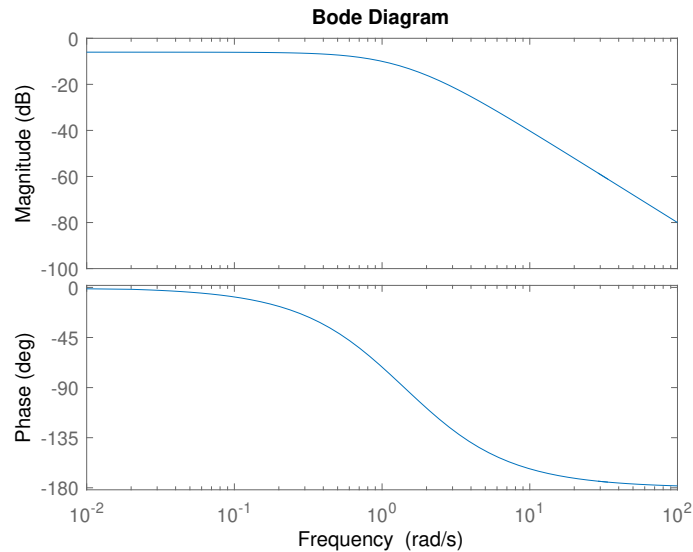
Settling time: as $t_s \approx \frac{3}{\sigma}$, minimum is achieved when the roots are farthest away from $j\omega$ -axis. In this situation again $s = -2 - \sqrt{2}$ and $t_s \approx \frac{3}{3.4} \approx 0.88$. Again in this case $K = 3 + 2\sqrt{2}$.

2. Consider the plant with transfer function

$$G(s) = \frac{1}{(s + 1)(s + 2)}$$

a) Sketch the Bode plot of $G(s)$ (for $K = 1$). Explain what rules you used to plot it.

Solution:



For magnitude plot, it starts from $|G(0)| = \frac{1}{2} \approx -6dB$. slop starts at 0, goes down by 1 at each of the two breakpoints ($\omega = 1, 2$) which correspond to stable poles

For phase plot, it starts at 0° , goes down by 90° after each breakpoint.

- b) Find a value of K such that for the Bode plot of $KG(s)$ the crossover frequency becomes $\omega_c = 3$.

Solution:

Currently $|G(3)| = \left| \frac{1}{|3j+1||3j+2|} \right| = \frac{1}{\sqrt{130}} \approx \frac{1}{11.4}$, So we need $K = 11.4$

c) With K fixed at the value you found in part b), suppose you want to use a lead or lag controller to increase the phase margin by at least 30° . Explain which controller type (lead or lag) you would choose and how you would design it. (If finding exact parameters of the controller by hand is difficult, describe the procedure you would follow if Matlab were available to you.)

Solution:

Use lead controller $D_1(s) = \frac{s+z_1+1}{s+p_1}$ with $\sqrt{z_1 p_1} \approx \omega_c = 3, p_1 > z_1$. First attempt is to set $p_1 = 9, z_1 = 1$, which should give enough increase (more than 30°) in PM. Consequently the controller is found to be $D_1(s) = \frac{s+1}{s+9}$. Notice that lead controller will potentially shift up ω_c and more iterations may be needed.

d) With K from part b) and the lead or lag controller from part c) fixed, suppose you want to add another lead or lag controller to ensure steady-state tracking of constant references within 10%. Explain which controller type (lead or lag) you would choose and how you would design it.

Solution:

Here lag controller is used: $D_2(s) = \frac{s+z_2}{s+p_2}$, with $p_2 < z_2$.

$$e(\infty) = \frac{1}{1 + KD_1(s)D_2(s)G(s)} \Big|_{s=0} = \frac{1}{1 + 11.4 \cdot 1 \cdot \frac{z_2}{p_2} \cdot \frac{1}{2}} \leq 10 \Rightarrow \frac{z_2}{p_2} \geq 1.58$$

In addition, z_2, p_2 needs to be sufficiently small so not to mess up with ω_c . Hence pick $z_2 = 0.2, p_2 = 0.1$. Therefore the lag controller used is $D_2(s) = \frac{s+0.2}{s+0.1}$.