Instruction: Please write your answers and explanations clearly and concisely, and try to fit them in the space provided (use the back of the page if necessary). Include at least a brief justification with each answer. You don't need to submit your scratch notes.
All questions on this exam refer to the plant transfer function

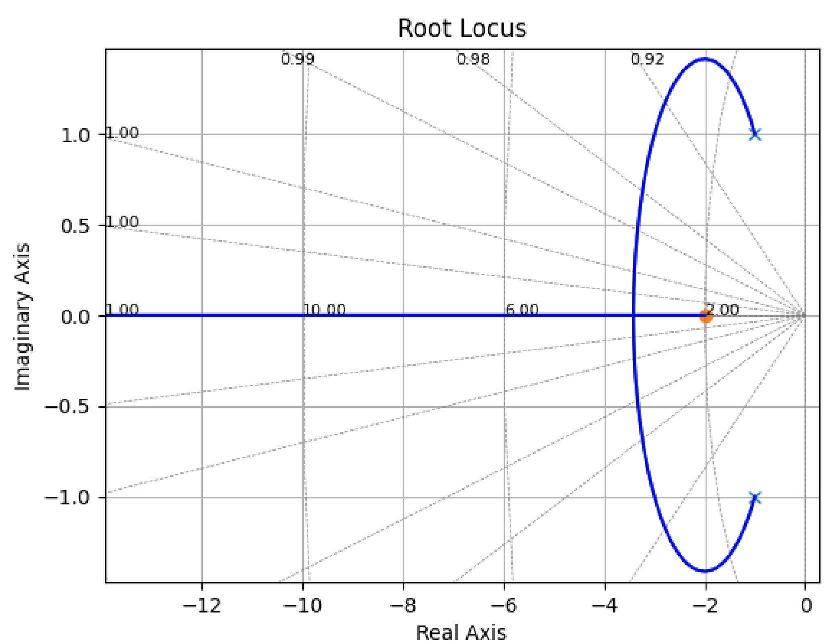
$$G(s) = \frac{s+2}{(s+1)^2+1}$$

Also, the closed-loop system under a feedback gain K or a more general controller always corresponds to the standard negative unity feedback configuration.

1. Sketch the positive root locus for G(s). Based on the rules discussed in class, be sure to describe the following: the number of branches, where they start and where they are going; the real-axis portion of the root locus; $j\omega$-axis crossings (if any); points of multiple roots (if any). You don't need to calculate any departure or arrival angles.

Solution:

- Start point: 2 branches start at open-loop poles, $s = -1 \pm j$.
- End point: 1 branch goes to open-loop zero $s = -2$ and 1 branch goes to ∞.
- Real axis: $(-\infty, -2)$.
- Closed-loop characteristic equation $s^2 + (k+2)s + 2(k+1) = 0$ is stable $\forall k > -1$, thus there is no $j\omega - axis$ crossing.
- Multiple roots: simplifying $(s+2)(2s+2) = s^2+2s+2$ gives $s^2+4s+2 = 0$. $s = -2 \pm \sqrt{2}$. $-2+\sqrt{2}$ is not on RL. $-2-\sqrt{2}$ is point of multiple roots.



2. Sketch the Bode plots (magnitude and phase) for G(s) (with K = 1). Explain how you arrived at your plots. Don't worry too much about exact behavior close to the breakpoints.

Solution:

Break point for $\frac{1}{s^2+2s+2}$ is $\omega = \sqrt{2}$.

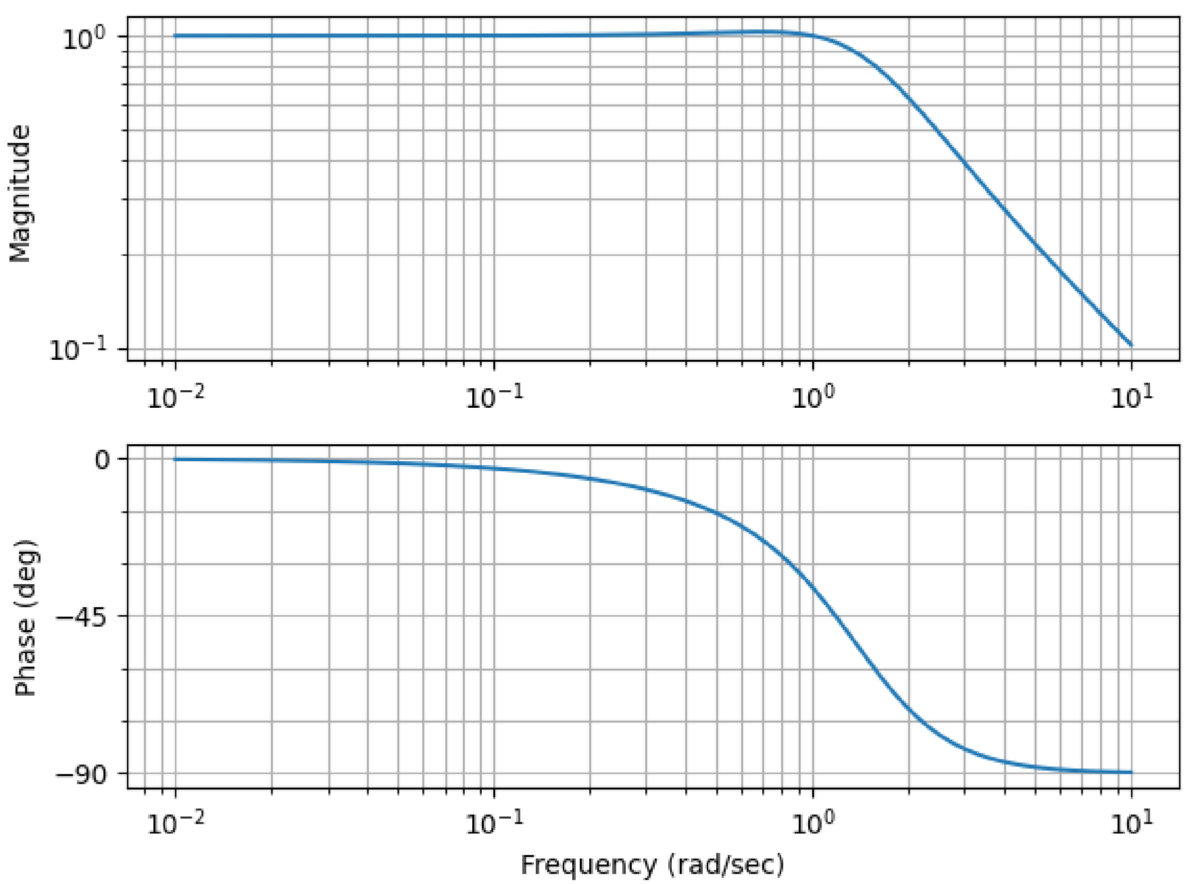
Break point for $s + 2$ is $\omega = 2$.

Magnitude graph:

- Starts at 1 with slope 0.
- After $\omega = \sqrt{2}$, slope decreases to -2.
- After $\omega = 2$, slope increases to -1.

Phase graph:

- Starts at 0°.
- After $\omega = \sqrt{2}$, goes towards $-180°$.
- After $\omega = 2$, goes back up to 90°.
- Cumulative effect is phase goes from 0° to$-90°$.



3.

a) Suppose you want to find a feedback gain that is a positive integer ($K = 1, 2, 3, \dots$) such that the closed-loop system has overshoot M_p as small as possible and settling time ts is as fast possible. Explain what value of K you would pick and why.

Solution:

Referring to the root locus in Problem1. The best location for closed-loop poles is at $-2-\sqrt{2}$, the point of multiple roots, because negative real roots means no overshoot. For large K, one pole moves right towards -2, which is not good for settling time.
Thus, closed-loop characteristic equation is $(s+2+\sqrt{2})^2$ and it also equals to $s^2+(K+2)s+2(K+1)$. We finally get $K = 2+2\sqrt{2}$ and the nearest integer is $K = 5$.

b) Let K be the gain you found in part a). For this K, suppose that you want to calculate the phase margin (PM). To do this, you can rely on the plots you made earlier, but you cannot create any new plots. Imagining that you could get exact coordinates of any point on the above plots (as you would in Matlab by clicking on a point), explain how you would obtain the exact value of PM. Also, approximately what answer do you expect to get?

Solution:

$K = 5$. To find PM, crossover frequency ω_c is needed. Due to definition of ω_c, $|KG(j\omega_c)| = 1$, $|G(j\omega_c)| = \frac{1}{5}$. Thus, we need to find ω, where magnitude is $\frac{1}{5}$.
At ω_c, measure phase and determine how much it differs from $-180°$. And the difference is PM. From the plots, it appears that PM=90°.

c) With K still the same as in part a), suppose you want to add a lead or lag controller to get steady-state tracking of constant references within 2%, without affecting the phase margin. Explain which controller type (lead or lag) you would choose and how you would design it.

Solution:

With $K = 5$ and without lead/lag controller, steady-state tracking error is $e(\infty) = \frac{1}{1+KG(0)} = \frac{1}{6}$.
With $K = 5$ and with lead/lag controller,$D(s) = \frac{s+z}{s+p}$, steady-state tracking error is $e(\infty) = \frac{1}{1+KD(0)G(0)} = \frac{1}{1+5\frac{z}{p}}$. To meet the requirement that steady-state tracking error is within 2%, we need $\frac{z}{p} = 10$. It is lag.
Not to disturb PM, z & p should be much smaller than ω_c. For $K = 5$, $\omega_c > 1$. Thus, z=0.1 and p=0.01 are answers that work.