
YOUR NAME: ______________________________

Instructions: Please write your answers clearly and concisely, and try to fit them in the space provided (use the back of the page if necessary). You don't need to submit your scratch notes.

1. Consider the plant transfer function

$$G(s) = \frac{s+1}{s^2+2s+3}$$

a) Draw an all-integrator diagram for this system.

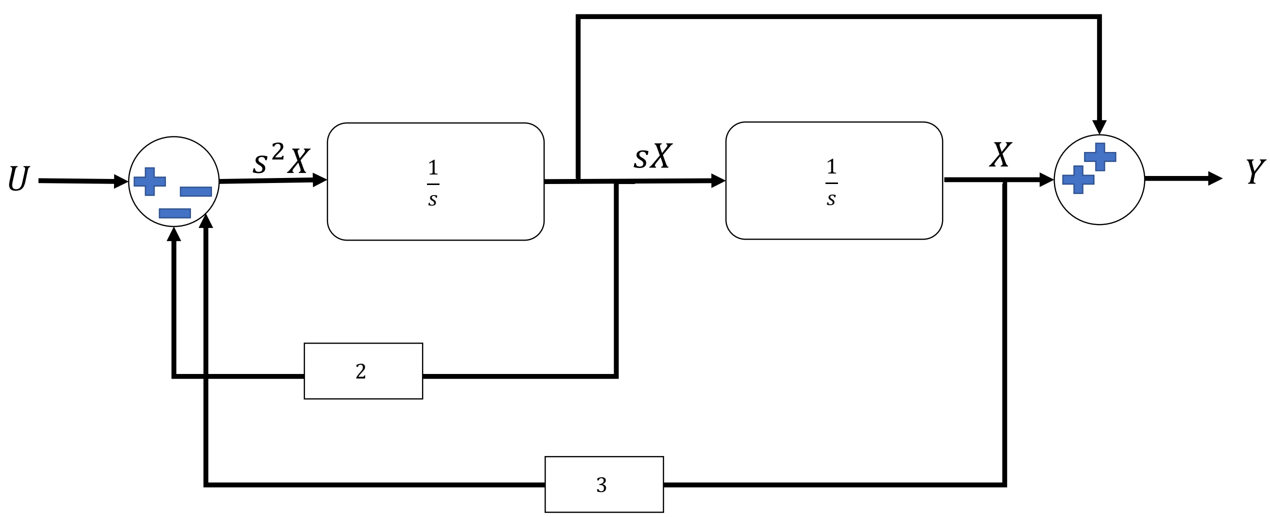

Figure 1: All integrator block diagram for the transfer function $G(s) = \frac{s+1}{s^2+2s+3}$

b) Write down a state-space model, in the form $\dot{x} = Ax + Bu$, $y = Cx$, that matches the diagram you obtained in part a). We have the following ODE for the given transfer function $G(s)$.

$$\begin{aligned} \ddot{x} &= -3x - 2\dot{x} + u \\ y &= x + \dot{x} \end{aligned} \iff \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -3x_1 - 2x_2 + u \\ y &= x_1 + x_2 \end{aligned}$$

In the state-space form we can similarly write these equations as follows,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

c) Is the above system stable? Explain.
Yes, For a second order quadratic polynomial to have roots in the LHP, it is necessary and sufficient that its coefficients is > 0.

d) What is the steady-state response of the above system to the unit step input? (As in class, by "steady-state response" we mean the component of the output $y(t)$ that persists after the transients have died down.)
Since, the system is stable we can apply the final value theorem (FVT) to look at the DC gain

$$\begin{aligned}\text{DC Gain} &= \lim_{s\to 0} sG(s)U(s) \quad U(s) = \frac{1}{s}(\text{ Unit Input}) \\ &= \lim_{s\to 0} G(s) = \frac{1}{3}\end{aligned}$$

e) What is the steady-state response of the above system to the input $u(t) = \cos t$?
Using the frequency response formula at Steady State, the response to the signal $u(t) = \cos \omega t$ would be $y(t) = M\cos(\omega t + \phi)$ Here, $M = |G(j\omega)|$ and $\phi = \angle G(j\omega)$ Thus, we get $|G(j\omega)| = \frac{1}{2}, \angle G(j\omega) = 0$. We thus get the final answer as follows,

$y(t) = \frac{1}{2}\cos t$

2. Now suppose that the system from Problem 1 is connected to a feedback gain $K > 0$, in the standard negative unity feedback configuration.

a) Draw a block diagram of the closed-loop system. (Note: you *don't* need to draw an all-integrator diagram; the diagram can have the transfer function $G(s)$ as a single block.) Mark the following signals in the diagram: R (reference), E (tracking error), U (control input), Y (output). Here, the closed loop

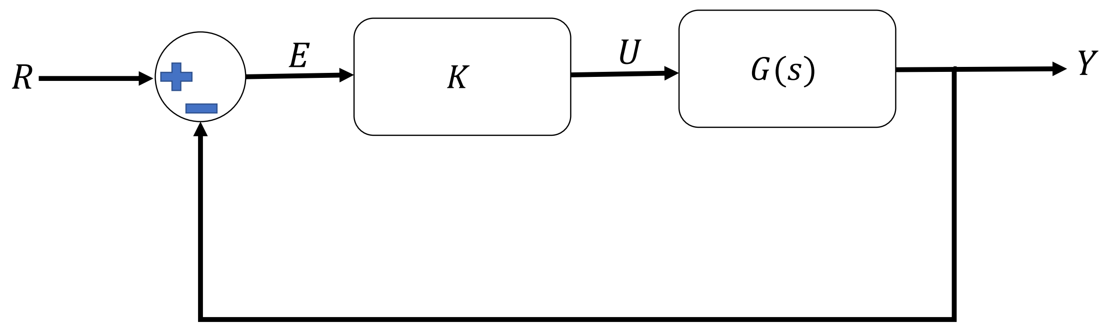

Figure 2: Block diagram for the closed loop system with controller K

transfer function of the system is

$$\frac{Y}{R} = \frac{KG(s)}{1 + KG(s)}$$

b) How high should the feedback gain K be so that the steady-state tracking error of the closed-loop system in response to constant references does not exceed 10%?

We can calculate the reference error transfer function $\frac{E}{R} = \frac{Y-R}{R}$ as follows,

$$\frac{E}{R} = \frac{1}{1+KG(s)} = \frac{s^2+2s+3}{s^2+(K+2)s+K+3}$$

We want to make sure that the DC gain is less than 10%.

$$\begin{aligned}\lim_{s\to 0} &= \lim_{s\to 0}\frac{s^2+2s+3}{s^2+(K+2)s+K+3} = \frac{3}{K+3} \leq 0.1 \\ K &\geq 27\end{aligned}$$

c) True or false: the settling time, t_s, for the closed-loop system can be made arbitrarily small by choosing the gain K large enough. Justify your answer.

False It is worth noting that the closed loop poles for the transfer function given in 2(a) is as follows,

$$\begin{aligned}\text{Roots} &= \frac{-(K+2) \pm \sqrt{(K+2)^2 - 4(K+3)}}{2} \\ &= \frac{-(K+2) \pm \sqrt{K^2-8}}{2}\end{aligned}$$

Looking at this equation we realize that as $K \to \infty$, both of the poles for the closed loop equation are real. With one of them going to $-\infty$. At the same time the other root goes to -1

$$\text{Roots} \;=\lim_{K\to\infty}\frac{-(K+2) \pm \sqrt{(K+2)^2 - 4(K+3)}}{2} = -\infty, -1$$

Thus, the rate limiting step is going to be -1. This limits our ability to make t_s arbitarily small.