Reading: FPE, Sections 5.1, 5.2.

Problems:

1. Consider the plant with transfer function $L(s) = \dfrac{1}{s^2+2s}$. Under the action of a constant feedback gain K, the closed-loop poles are the roots of the characteristic polynomial s^2+2s+K.

a) Draw the (positive) root locus. (Use the expression for the closed-loop poles in terms of K obtained via the quadratic formula.)

b) Consider the settling time spec $t_s \le 4$. Give some value (or range of values) of K for which the closed-loop system meets this spec. Justify your choice. Show the corresponding pole locations on the root locus.

c) Consider the rise time spec $t_r \le 1$. Give some value (or range of values) of K for which the closed-loop system meets this spec. Justify your choice. Show the corresponding pole locations on the root locus.

d) Consider the overshoot spec $M_p \le 0.1$. Give some value (or range of values) of K for which the closed-loop system meets this spec. Justify your choice. Show the corresponding pole locations on the root locus.

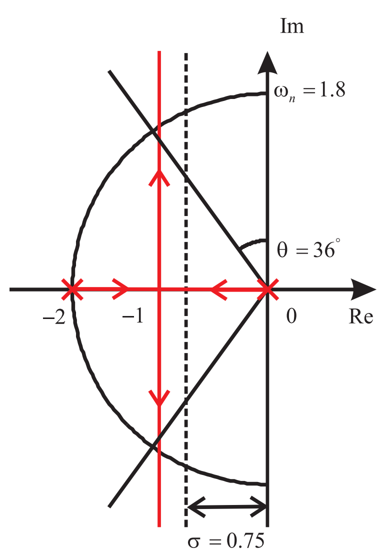
e) Suppose that it is desired to place the closed-loop poles at $-1 \pm j$. Find the value of K that will achieve this, using the characteristic equation $s^2+2s+K=0$ *but without using the quadratic formula*. (In other words, you should find a way of doing this that would also work for a higher-order example.)

Solution:

a) Root Locus:

$$\# \text{ Poles} = n = 2, \#\text{zeros} = m = 0$$

$$\Rightarrow \begin{cases} \# \text{ of asymptotes} = n - m = 2 \\ \alpha = \dfrac{\Sigma p_i - \Sigma z_i}{n-m} = \dfrac{-2}{2} = -1 \quad \text{(center of asymptotes)} \\ \phi = \dfrac{(2k+1)\pi}{n-m} = \pm\dfrac{\pi}{2} \quad \text{(angle of asymptotes)} \end{cases}$$



b) $t_s \leq 4 \Rightarrow \dfrac{3}{\sigma} \leq 4 \Rightarrow \sigma \geq 0.75$
To satisfy this, we need $K > 2 \times 0.75 - (0.75)^2$ or $K > 0.9375$ to make sure that both poles are lying on the left of ($\sigma = 0.75$) line (dashed).

c) $t_r \leq 1 \Rightarrow \dfrac{1.8}{\omega_n} \leq 1 \Rightarrow \omega_n \geq 1.8$
To make sure the poles are outside the $\omega_n = 1.8$ circle, we need

$$K \geq \omega_n^2 \Rightarrow K \geq 3.24$$

d) $M_p \leq 0.1$

$$e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}} \leq 0.1 \Rightarrow \begin{cases} \frac{\pi\xi}{\sqrt{1-\xi^2}} \geq 2.3 \Rightarrow \xi \geq 0.6 \\ \text{or} \\ \pi \tan\theta \geq 2.3 \Rightarrow \theta \geq 36^\circ \end{cases}$$

but in characteristic eq: $s^2 + 2s + K \approx s^2 + 2\xi\omega_n s + \omega_n^2$

$$\Rightarrow \left.\begin{array}{l} \omega_n^2 = K \\ \xi = \sqrt{\frac{1}{K}} \\ \xi \geq 0.6 \end{array}\right\} \Rightarrow \sqrt{\frac{1}{K}} \geq 0.6 \Rightarrow K \leq 2.78$$

e) Closed loop poles: $s = -1 \pm j$
$\Rightarrow$ characteristic equation: $(s-1+j)(s-1-j) = s^2 + 2s + 2 \equiv s^2 + 2s + K \therefore K = 2$
(**) In general, we can also check the gain condition for root Locus (future lectures).

2. Consider the following transfer functions:

$$1)\ L(s) = \frac{1}{s(s^2+4s+8)} \qquad\qquad 2)\ L(s) = \frac{s}{(s-1)(s+1)^2}$$

For each one of these, do the following:

a) Mark the zeros and poles on the s-plane and use Rule 2 from class to plot the real-axis part of the root locus.

b) Use the phase condition from class to test whether or not the point $s = j$ is on the root locus. If you run into "non-obvious" angles, *estimate* rather than *calculate* them, this should be enough.

c) Apply Rules 3 and 4 to determine asymptotes and departure and arrival angles. Plot the root locus branches based on this information.

d) Apply Rule 5 to determine imaginary-axis crossings (if any), and complete the (positive) root locus by using Rule 6 to check for multiple roots.

e) Plot the (positive) root locus using the MATLAB `rlocus` command.

f) Repeat items a)–e) for the *negative* root locus.

Turn in your MATLAB plots as well as hand sketches of root loci along with all accompanying calculations and explanations.

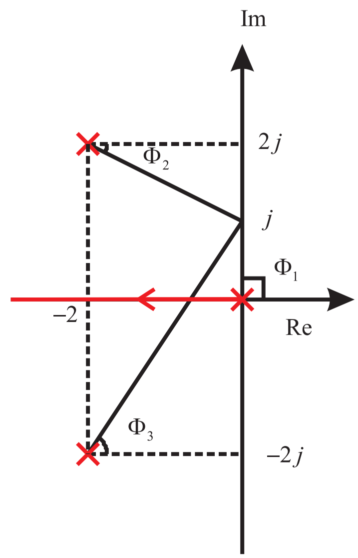
Solution:

1) a) Poles: $p_1 = 0, p_{2,3} = -2 \pm 2j$
$n = 3, m = 0 \Rightarrow 3$ asymptotes
The negative side of real axis is on loci.

b) Phase condition check

$$\phi_1 + \phi_2 + \phi_3 \overset{?}{=} \pi(2k+1)$$
$$\phi_1 = \pi/2, \phi_2 = \tan^{-1}\left(\frac{-1}{2}\right), \phi_3 = \tan^{-1}\left(\frac{3}{2}\right)$$
$$\Rightarrow \phi_1 + \phi_2 + \phi_3 < \pi \quad \text{(because } \phi_3 + \phi_2 < \pi/2\text{)}$$

$\therefore s = j$ is NOT on loci.



c) Angle of asymptotes:

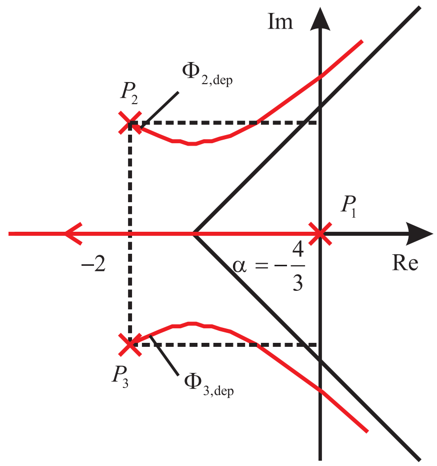
$$\frac{(2k+1)\pi}{n-m} = \pm\frac{\pi}{3} \text{ and } \pi$$

Center of asymptotes:

$$\alpha = \frac{\sum p_i - \sum z_i}{n-m} = -\frac{4}{3}$$

Departure angles:

$$\phi_{1,dep} = \pi$$
$$\phi_{2,dep} = \pi - \frac{3\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$$
$$\phi_{3,dep} = -\phi_{2,dep} = \frac{\pi}{4}$$



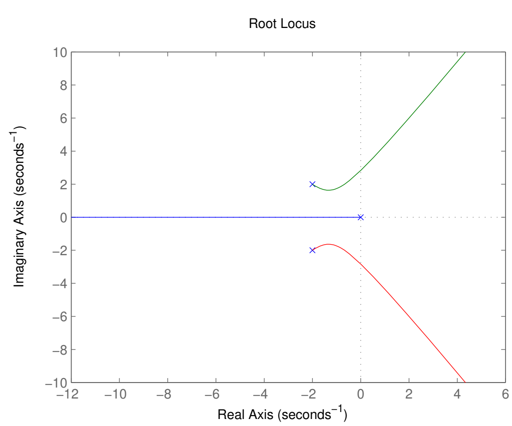
d) $j\omega$-crossings:

$$s(s^2+4s+8)+K\big|_{s=j\omega}=0$$
$$\Rightarrow -j\omega^3-4\omega^2+8j\omega+K=0$$
$$\Rightarrow K=4\omega^2,\ \omega^3-8\omega=0\Rightarrow \omega=0,\pm\sqrt{8},\ K=0,32$$

Multiple roots:

$$b(s)\frac{da(s)}{ds}-a(s)\frac{db(s)}{ds}=0$$
$$(3s^2+8s+8)-(s^3+4s^2+8s)\cdot 0=s^3+s^2-s-1=0$$
$$\Rightarrow s=-1.333\pm 0.9428 \text{ (not valid)}$$

e)



f) 1-a) Poles: $p_1=0, p_{2,3}=-2\pm 2j$
$n=3, m=0\Rightarrow 3$ asymptotes
The positive side of real axis is on loci.

1-b) Phase condition check

$$\phi_1+\phi_2+\phi_3\overset{?}{=}2k\pi$$
$$\phi_1=\pi/2,\phi_2=\tan^{-1}\left(\frac{-1}{2}\right),\phi_3=\tan^{-1}\left(\frac{3}{2}\right)$$
$$\Rightarrow \phi_1+\phi_2+\phi_3\neq 2k\pi \quad (\text{because } \phi_3+\phi_2<\pi/2)$$

$\therefore s=j$ is NOT on loci.

1-c) Angle of asymptotes:

$$\frac{2k\pi}{n-m} = \pm\frac{2\pi}{3} \text{ and } 0$$

Center of asymptotes: The same as positive root locus

Departure angles:

$$\phi_{1,dep} = 0$$

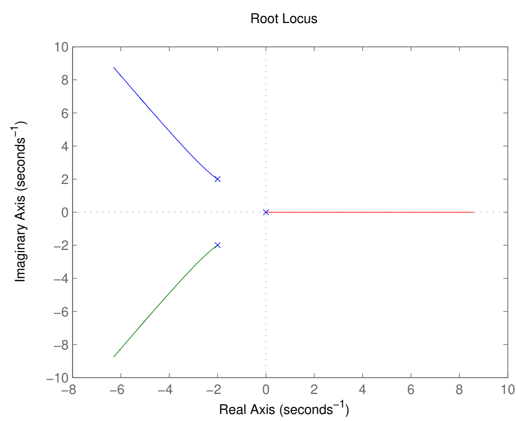
$$\phi_{2,dep} = 2\pi - \frac{3\pi}{4} - \frac{\pi}{2} = \frac{3\pi}{4}$$

$$\phi_{3,dep} = -\phi_{2,dep} = -\frac{3\pi}{4}$$

1-d) $j\omega$ - crossings: From d)-1), $\omega = 0, \quad K = 0$

Multiple roots: The same as positive root locus.

1-e)



2) a) Poles: $p_1 = 1, p_{2,3} = -1$, Zeros: $z_1 = 0$

$n = 3, m = 1 \Rightarrow 2$ asymptotes

The only part of real axis on Loci is $0 \leq \sigma \leq 1$.

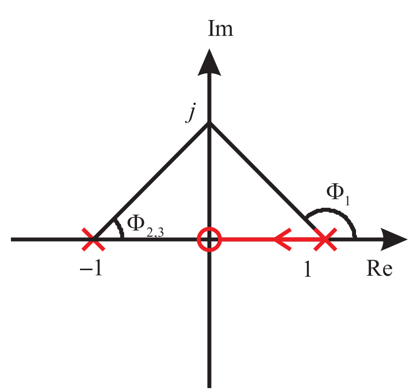
b)

$$\phi_1 + \phi_2 + \phi_3 - \psi_1 \stackrel{?}{=} \pi(2k+1)$$

$$\phi_1 = \frac{3\pi}{4}, \phi_2 = \phi_3 = \pi/4, \psi_1 = \pi/2$$

$$\frac{3\pi}{4} + \frac{\pi}{4} + \frac{\pi}{4} - \frac{\pi}{2} = \frac{3\pi}{4} \neq \pi$$

$\therefore s = j$ is NOT on loci.



c) Angle of asymptotes $= \pm\frac{\pi}{2}$
Center of asymptotes:

$$\alpha = \frac{\sum p_i - \sum z_i}{n-m} = -\frac{1}{2}$$

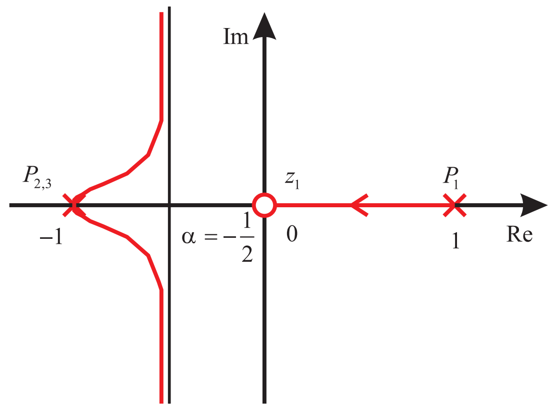
Departure and arrival angles:

$$\phi_{1,dep} = \pi$$

$$\psi_{1,arr} = 0$$

$$\phi_{2,dep} = \frac{1}{2}(\pi - 0 - 0) = \frac{\pi}{2}$$

$$\phi_{3,dep} = \frac{1}{2}(\pi - 2\pi) = -\frac{\pi}{2}$$



d) $j\omega$-crossings:

$$(s-1)(s+1)^2 + KS\big|_{s=j\omega} = 0$$

$$\Rightarrow s^3 + s^2 + (K-1)s - 1\big|_{s=j\omega} = 0$$

$$\Rightarrow -j\omega^3 - \omega^2 + j\omega(K-1) - 1 = 0$$

$$\Rightarrow \omega(\omega^2 - K + 1) = 0,\ \omega^2 = -1 \text{ (No solution)}$$

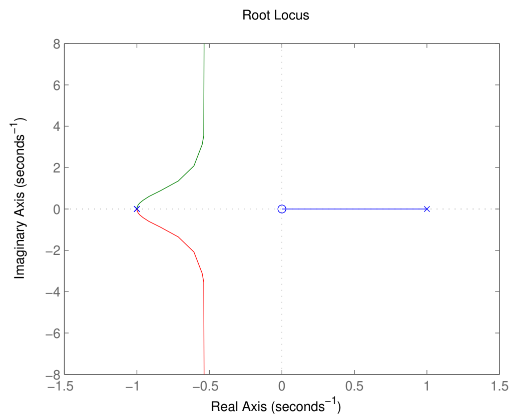
This can be checked by Routh method.
Multiple roots:

$$b(s)\frac{da(s)}{ds} - a(s)\frac{db(s)}{ds} = 0$$

$$s(3s^2 + 2s - 1) - (s^3 + s^2 - s - 1) = 2s^3 + s^2 + 1 = 0$$

$$\Rightarrow s = -1 \text{ (valid)},\ s = 0.25 \pm 0.6614j \text{ (not valid)}$$

e)



f) 2-a) Poles: $p_1 = 1, p_{2,3} = -1$, Zeros: $z_1 = 0$

$n = 3, m = 1 \Rightarrow 2$ asymptotes

The part of real axis on Loci is $[-\infty, 0]$ and $[1, \infty]$.

2-b)

$$\phi_1 + \phi_2 + \phi_3 - \psi_1 \stackrel{?}{=} 2k\pi$$

$$\phi_1 = \frac{3\pi}{4}, \phi_2 = \phi_3 = \pi/4, \psi_1 = \pi/2$$

$$\frac{3\pi}{4} + \frac{\pi}{4} + \frac{\pi}{4} - \frac{\pi}{2} = \frac{3\pi}{4} \neq 2k\pi$$

$\therefore s = j$ is NOT on loci.

2-c) Angle of asymptotes $= 0,\ \pi$

Center of asymptotes: The same as positive root locus.

Departure and arrival angles:

$$\phi_{1,dep} = 0$$

$$\psi_{1,arr} = \pi$$

$$\phi_{2,dep} = \frac{1}{2}(2\pi + \pi - \pi - 0) = \pi$$

$$\phi_{3,dep} = -\phi_{2,dep} = -\pi$$

2-d) $j - \omega$ crossings: From d)-2), no solution.

Multiple roots: From d)-2), $s = -1$.

2-e)