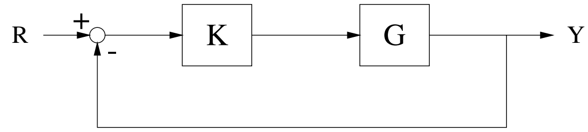
ECE 486 **PROBLEM SET 4 - SOLUTION** **Fall 2023**

Reading: FPE, Sections 4.1-4.3 (the material not discussed in class is optional).

Problems:

1. Consider the following feedback system, where K is a constant gain and $G(s) = \dfrac{1}{s^3 + 2s^2 + 2s + 1}$:



Using Routh's criterion, show that for $-1 < K < 3$ the system is stable but for $K \geq 3$ the system is unstable. (This illustrates the destabilizing effect of feedback when the gain is too high.)

Solution:

$$G_{RY} = \frac{KG}{1 + KG} = \frac{K}{s^3 + 2s^2 + 2s + (K+1)}$$

Routh array for the denominator of the closed-loop transfer function

$$\begin{array}{lccc} s^3: & 1 & 2 & \\ s^2: & 2 & K+1 & \\ s^1: & \frac{4-(K+1)}{2} & & \rightarrow \frac{3-K}{2} \\ s^0: & K+1 & & \end{array}$$

for stability, we need to make sure that there is no sign change in the first column of the above array. Thus,

$$\frac{3-K}{2} > 0, \quad K + 1 > 0 \qquad \Rightarrow -1 < K < 3.$$

2. The goal of this exercise is to compare sensitivity of open-loop and closed-loop control with respect to errors in the *control gain*. Consider the DC motor model discussed in class, with no disturbance ($\tau_L = 0$). Let the control gain sensitivity be defined as follows: when the controller gain changes from K to $K + \delta K$ and, as a result, the steady state gain (DC gain) of the overall system changes from T to $T + \delta T$, we define $S_K = \dfrac{\delta T / T}{\delta K / K}$. (The motor gain A remains fixed here.)

a) Compute the sensitivity S_K in the open-loop case, starting from the nominal values $K_{ol} = 1/A$ and $T_{ol} = 1$.

b) Compute the sensitivity S_K for a feedback gain K_{cl}, using the approximate formula $\delta T = \dfrac{dT}{dK}\delta K$ and the fact that the nominal system gain is, as derived in class, $T_{cl} = \dfrac{AK_{cl}}{1 + AK_{cl}}$.

Hint: your final answers in a) and b) should be the same as the ones derived in class for sensitivity to errors in the motor gain A.

Solution:

a)

$$T_{ol} = 1, \quad T_{ol} + \delta T_{ol} = A(K_{ol} + \delta K_{ol}) = A \times \frac{1}{A} + A \times \delta \left(\frac{1}{A}\right) = T_{ol} + A \times \delta \left(\frac{1}{A}\right)$$

$$\Rightarrow \delta T_{ol} = A\delta \left(\frac{1}{A}\right) = A\delta K_{ol}$$

$$\Rightarrow S_k = \frac{\frac{\delta T_{ol}}{T_{ol}}}{\frac{\delta K_{ol}}{K_{ol}}} = \frac{\frac{A\delta K_{ol}}{1}}{\frac{\delta K_{ol}}{1/A}} = 1$$

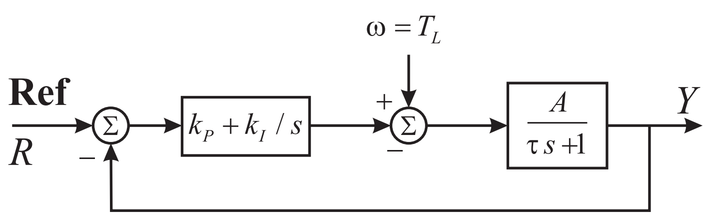
b)

$$\frac{\delta T_{cl}}{\delta K_{cl}} = \frac{A}{(1 + AK_{cl})^2}$$

$$S_{K_{cl}} = \frac{\delta T_{cl}}{\delta K_{cl}} \times \frac{K_{cl}}{T_{cl}} = \frac{A}{(1 + AK_{cl})^2} \times \frac{K_{cl}}{\frac{AK_{cl}}{1+AK_{cl}}} = \frac{1}{1 + AK_{cl}}$$

3. Suppose that the DC motor discussed in class is connected in feedback with a PI controller $k_P + k_I/s$. (This refers to the standard feedback control configuration, where the input to the controller is $e = r - y = \omega_{ref} - \omega_m$.) Write down the full transfer function of the closed-loop system in the presence of load/disturbance τ_L. (For $k_I = 0$ this should match what we derived in class for constant feedback gain.) Is it true that by proper choice of gains k_P and k_I we can achieve arbitrary pole placement as well as perfect constant reference tracking and constant disturbance rejection in steady state? Justify your answer.

Solution:



Transfer function:

$$Y = [(k_P + k_I/s)(R - Y) + T_L] \left(\frac{A}{\tau s + 1}\right)$$

which gives:

$$Y = \frac{A(k_P s + k_I)}{\tau s^2 + (Ak_P + 1)s + Ak_I} R + \frac{As}{\tau s^2 + (Ak_P + 1)s + Ak_I} T_L = G_{RY}R + G_{WY}T_L$$

by the proper choice of k_P and k_I, we can assign the poles of transfer function from input (G_{RY}) by choosing two parameters for two poles.

Constant reference tracking:

$$E = 1 - G_{RY} = \frac{\tau s^2 + (Ak_P + 1)s + Ak_I - Ak_P s - Ak_I}{\tau s^2 + (Ak_P + 1)s + Ak_I} = \frac{\tau s^2 + s}{\tau s^2 + (Ak_P + 1)s + Ak_I}$$

E has a zero DC gain. Also, its final value for constant r ($r(s) = \frac{\alpha}{s}$):

$$e = \lim_{s \to 0} s \frac{\tau s^2 + s}{\tau s^2 + (k_P A + 1)s + Ak_I} \times \frac{\alpha}{s} = 0$$

$\therefore$ Assuming proper choice for the poles (LHP), PI controller will accomplish perfect tracking.

Constant disturbance rejection: G_{WY} has a zero DC gain. Also, its final vale for constant disturbance $W(s) = \frac{\beta}{s}$ is:

$$y_w(\infty) = \lim_{s\to 0} sG_{WY}(s)W(s) = \lim_{s\to 0} s\frac{As}{\tau s^2 + (Ak_P + 1)s + Ak_I} \times \frac{\beta}{s}$$

$\therefore$ Assuming proper choice of poles (LHP), constant disturbances will be rejected perfectly.

4. Consider again the standard feedback configuration like the one in Problem 1, but with $K(s)$ and $G(s)$ unknown transfer functions. Suppose that the transfer function from R to Y is $\dfrac{Y(s)}{R(s)} = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ for some $\zeta, \omega_n > 0$.

a) Based on this information, find the forward gain $K(s)G(s)$.

b) Determine the system type and discuss what it implies about the system's steady-state tracking ability.

Solution:

a)

$$\frac{Y}{R} = \frac{K(s)G(s)}{1 + K(s)G(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2} = \frac{\omega_n^2}{s(s + 2\zeta\omega_n) + \omega_n^2} = \frac{\frac{\omega_n^2}{s(s+2\zeta\omega_n)}}{1 + \frac{\omega_n^2}{s(s+2\zeta\omega_n)}}$$

$$\therefore K(s)G(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$$

b) The system has one pole at the origin. Therefore, the system is a Type 1 system.

$$k_p = \lim_{s\to 0} K(s)G(s) = \lim_{s\to 0} \frac{\omega_n^2}{s(s + 2\zeta\omega_n)} = \infty \qquad \Rightarrow \frac{1}{1 + k_p} = 0$$

The system follows constant references (step) without error.

$$k_v = \lim_{s\to 0} sK(s)G(s) = \lim_{s\to 0} s\frac{\omega_n^2}{s(s + 2\zeta\omega_n)} = \frac{\omega_n}{2\zeta} \qquad \Rightarrow \frac{1}{k_v} = \frac{2\zeta}{\omega_n}$$

The system follows ramp references with constant error $\frac{2\zeta}{\omega_n}$.

$$k_a = \lim_{s\to 0} s^2K(s)G(s) = \lim_{s\to 0} s^2\frac{\omega_n^2}{s(s + 2\zeta\omega_n)} = 0 \qquad \Rightarrow \frac{1}{k_a} = \infty$$

The system cannot follow parabola references.