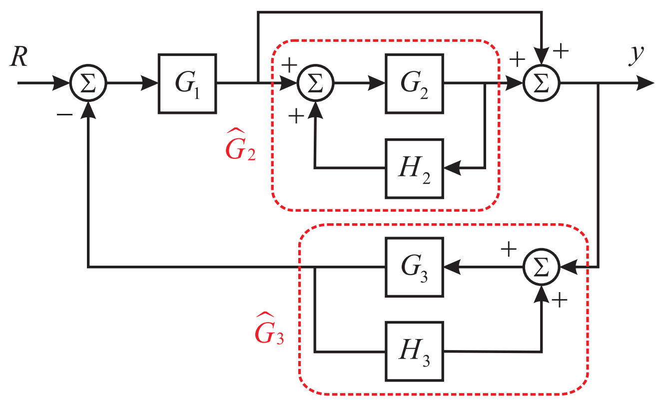
Reading: FPE, Sections 3.3-3.6.

Problems: (unless otherwise noted, you can use a calculator/computer to arrive at numerical answers)

1. Using techniques for block diagram reduction discussed in class, find the transfer functions of the systems a) and d) shown in FPE, Figure 3.54 on page 158 (6th edition), Figure 3.45 on page 154 (5th edition), Figure 3.55 on page 187 (4th edition). Of the four diagrams shown, only do the first and the last.

Solution:

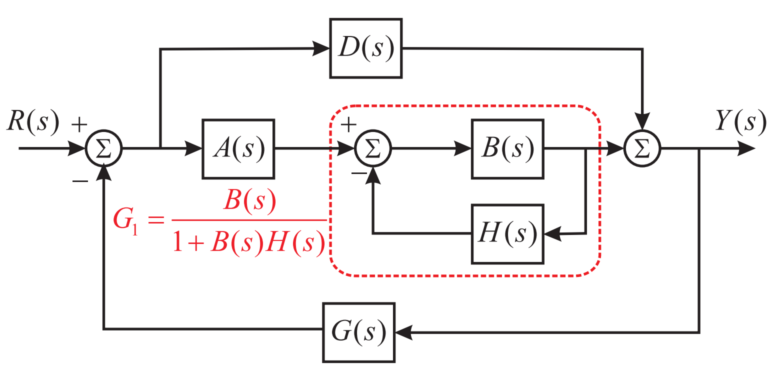
a)



$$\hat{G}_2 = \frac{G_2}{1 - G_2 H_2}, \quad \hat{G}_3 = \frac{G_3}{1 - G_3 H_3}$$

$$\begin{aligned} G_{RY} &= \frac{G_1(1+\hat{G}_2)}{1 + G_1(1+\hat{G}_2)\hat{G}_3} \\ &= \frac{G_1(1 - G_2H_2 + G_2)(1 - G_3H_3)}{(1 - G_2H_2)(1 - G_3H_3) + G_1(1 - G_2H_2 + G_2)G_3} \end{aligned}$$

d)



$$G_1 = \frac{B(s)}{1 + B(s)H(s)}$$

$$\begin{aligned} G_{RY} &= \frac{D + AG_1}{1 + G(D + AG_1)} \\ &= \frac{D + \frac{AB}{1+BH}}{1 + G\left(D + \frac{AB}{1+BH}\right)} \\ &= \frac{D + DBH + AB}{1 + BH + GD + GDBH + GAB} \end{aligned}$$

2. Consider the transfer function $H(s) = \dfrac{1}{s^2 + s + 1}$.

a) Suppose that you are given the following time-domain specs: $t_r \leq 2.5$, $t_s \leq 8$. Plot the admissible pole locations in the s-plane corresponding to these two specs. Does the given system satisfy these specs?

b) Suppose that *in addition to* the specs from a), we have the following spec on the overshoot: $M_p \leq 1/e^2$. Plot the admissible pole locations in the s-plane corresponding to all three specs. Does the given system satisfy the new spec?

c) Now suppose that you are given the two specs from a) *plus* the following spec on the peak time: $t_p \leq 4$ (*instead of* the overshoot spec). Plot the admissible pole locations in the s-plane corresponding to these three specs. (Pole locations for peak time were not discussed in class, so you need to derive this.) Does the given system satisfy the new spec?

Solution:

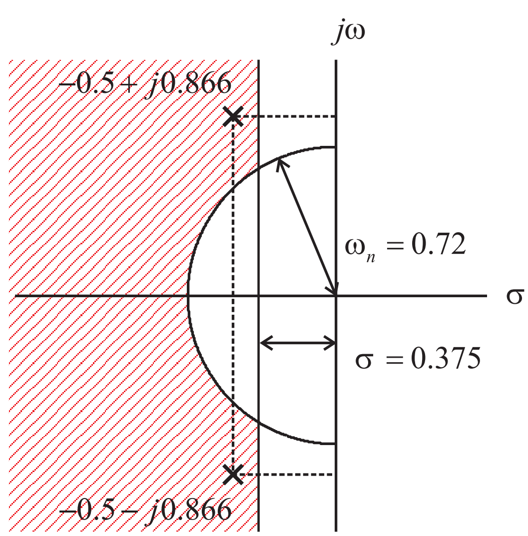
a)

$$t_r \leq 2.5 \Rightarrow \frac{1.8}{\omega_n} \leq 2.5 \Rightarrow \omega_n \geq 0.72$$

Considering 5% settling time:

$$t_s = \frac{3}{\sigma} \leq 8 \Rightarrow \sigma \geq \frac{3}{8} = 0.375$$

Poles of $H(s)$: $p_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}j$. We see that specs of part a) are satisfied since poles are in admissible shaded area.

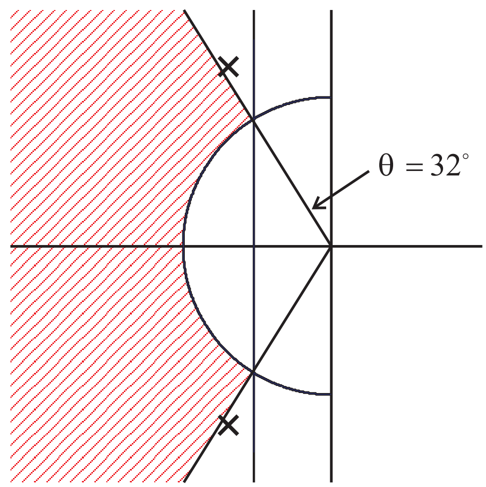


b)

$$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \leq e^{-2} \Rightarrow \frac{\zeta\pi}{\sqrt{1-\zeta^2}} \geq 2$$

$$\theta = \sin^{-1}\zeta = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right) \geq \tan^{-1}\left(\frac{2}{\pi}\right) = 32^\circ$$

Now the admissible region is the double shaded area and the poles do not lie in it. Therefore, the overshoot spec is not satisfied.

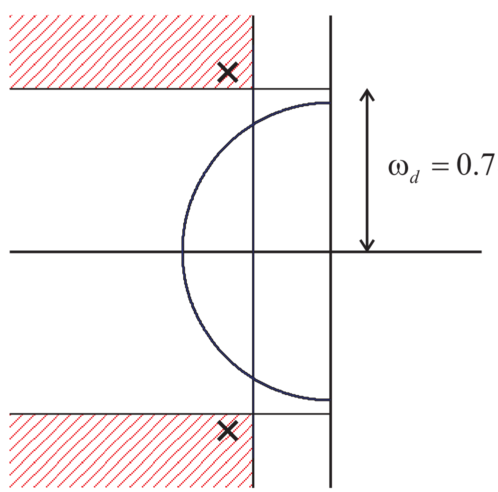


c)

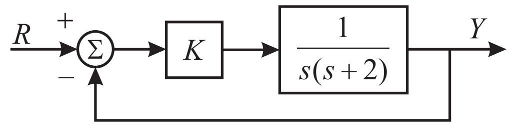
$$t_p = \frac{\pi}{\omega_d} \leq 4 \Rightarrow \omega_d \geq \frac{\pi}{4} = 0.785$$

The specs are satisfied but note that $t_p \leq 4$ overrules $t_r \leq 2.5$.

Note: in part a), we used 5% settling time. It can be easily verified that if one uses 1% settling time, even the first specs are not satisfied.



3. For the feedback system shown in the diagram



determine the range of proportional gains K so that the overshoot of the closed-loop system (in response to the unit step reference input) is no more than 10%.

Solution:

$$G_{RY} = \frac{K}{s^2 + 2s + K}$$

$$\Rightarrow \ \omega_n = \sqrt{K}, \ \zeta = \frac{1}{\sqrt{K}} \text{ or } K = \frac{1}{\zeta^2}$$

$$M_p \leq 10\% \Rightarrow \zeta \geq 0.59$$

$$\therefore 0 < K \leq 2.86$$

4. The purpose of this exercise is to illustrate that a system with a pair of poles on the imaginary axis is not stable because it gives unbounded response to a sinusoidal signal that matches the system's natural frequency (this phenomenon is known as *resonance*). Consider the transfer function

$$G(s) = \frac{1}{s^2 + \omega_n^2}, \qquad \omega_n > 0$$

and the input $u(t) = \sin(\omega t)$, $\omega > 0$.

a) Derive the response of the system, using both the partial fractions method and the frequency response formula. What happens to this response as ω approaches ω_n? (We know that the two methods give different answers, the first one being more precise in terms of the transient response, but qualitatively they should lead to the same conclusion here.)

b) To characterize the response more precisely, compute it using the convolution integral $y(t) = \int_0^t h(t-\tau)u(\tau)d\tau$ (note: these are the correct limits of integration). Identify the term in your calculation that leads to unbounded response when $\omega = \omega_n$. You can use the trigonometric identities $\sin(a-b) = \sin a \cos b - \cos a \sin b$ and $(\sin a)^2 = (1 - \cos 2a)/2$.

Solution:

a)

$$u(t) = \sin(\omega t), \ \omega > 0 \ \Rightarrow \ U(s) = \frac{\omega}{s^2 + \omega^2}$$

$$Y(s) = G(s)U(s) = \frac{\omega}{(s^2 + \omega_n^2)(s^2 + \omega^2)}$$

PFE:

$$Y(s) = \frac{\omega}{\omega_n^2 - \omega^2}\left[\frac{1}{s^2 + \omega^2} - \frac{1}{s^2 + \omega_n^2}\right]$$

$$\Rightarrow y(t) = \frac{\omega}{\omega_n^2 - \omega^2}\left(\frac{\sin(\omega t)}{\omega} - \frac{\sin(\omega_n t)}{\omega_n}\right)$$

Calculating the limit of the above function (using L'Hopital Rule) when $\omega \to \omega_n$, it can be shown that

$$y(t) \Rightarrow \frac{\sin(\omega_n t) - \omega_n t\cos(\omega_n t)}{2\omega_n^2}$$

which grows linearly with t. So when $t \to \infty$, $y \to \infty$.

Frequency Response Method:

$$G(s) = \frac{1}{s^2 + \omega_n^2} \Rightarrow G(j\omega) = \frac{1}{(j\omega)^2 - \omega_n^2} = \frac{1}{\omega_n^2 - \omega^2}$$

$\Rightarrow |G(j\omega)| = \left|\frac{1}{\omega_n^2 - \omega^2}\right| \to \infty$ when $\omega \to \omega_n$.

b) For part b), I assume that the input frequency is equal to natural frequency of the system i.e. $\omega = \omega_n$

$$G(s) = \frac{1}{s^2 + \omega_n^2} \Rightarrow g(t) = \frac{1}{\omega_n}\sin(\omega t)$$

$$\begin{aligned}
y(t) &= u(t) * g(t) \\
&= \int_0^t \sin(\omega_n(t-\tau))\frac{\sin(\omega_n\tau)}{\omega_n}d\tau \\
&= \frac{1}{\omega_n}\int_0^t \frac{\cos(\omega_n(t-\tau) - \omega_n\tau) - \cos(\omega_n(t-\tau) + \omega_n\tau)}{2}d\tau \\
&= \frac{1}{2\omega_n}\int_0^t \cos(\omega_n(t-2\tau)) - \cos(\omega_n t)d\tau \\
&= \frac{1}{2\omega_n}\int_0^t \cos(\omega_n(t-2\tau))d\tau - \frac{\cos(\omega_n t)}{2\omega_n}\int_0^t d\tau \\
&= \frac{1}{4\omega_n^2}\left[\sin(\omega_n(t-2\tau))\right]_0^t + \frac{t\cos(\omega_n t)}{2\omega_n} \\
&= \frac{\sin(\omega_n t)}{2\omega_n^2} - \frac{t\cos(\omega_n t)}{2\omega_n}
\end{aligned}$$

$\Rightarrow y(t) = \frac{\sin(\omega_n t)}{2\omega_n^2} - \frac{t\cos(\omega_n t)}{2\omega_n}$, which grows linearly with t. So when $t \to \infty$, $y \to \infty$.

5. Determine whether or not the following polynomials have any RHP roots:

a) $s^4 + 8s^3 + 32s^2 + 80s + 100$ b) $s^5 + 5s^4 + 2s^3 - s^2 + 4s + 10$

(Computer use not allowed.)

Solution:

a) $s^4 + 8s^3 + 32s^2 + 80s + 100$

$$
\begin{array}{llll}
s^4: & 1 & 32 & 100 \\
s^3: & 8 & 80 & \\
s^2: & -\dfrac{\begin{vmatrix} 1 & 32 \\ 8 & 80 \end{vmatrix}}{8} & 100 & \rightarrow \frac{32\times 8-80}{8} = 22 \\
s^1: & -\dfrac{\begin{vmatrix} 8 & 80 \\ 22 & 100 \end{vmatrix}}{22} & & \rightarrow \frac{22\times 80-800}{22} = \frac{480}{11} \\
s^0: & 100 & &
\end{array}
$$

There is no sign change in the first column of the Routh array $\Rightarrow$ There are no RHP roots for the above polynomial.

b) $s^5 + 5s^4 + 2s^3 - s^2 + 4s + 10$
Checking Necessary condition: There is a sign change in the coefficients of the polynomial while none of the coefficients are missing $\Rightarrow$ There is definitely an RHP root.