

**Reading:** FPE, Sections 6.3, 6.4 (re-read), 7.1–7.4, notes on state-space control by Prof. Belabbas (‘legacy documents’ section of the class website).

**Problems:**

1. For the two plant transfer functions given below, use the Nyquist stability criterion to determine all values of the feedback gain  $K$  that stabilize the closed-loop system.

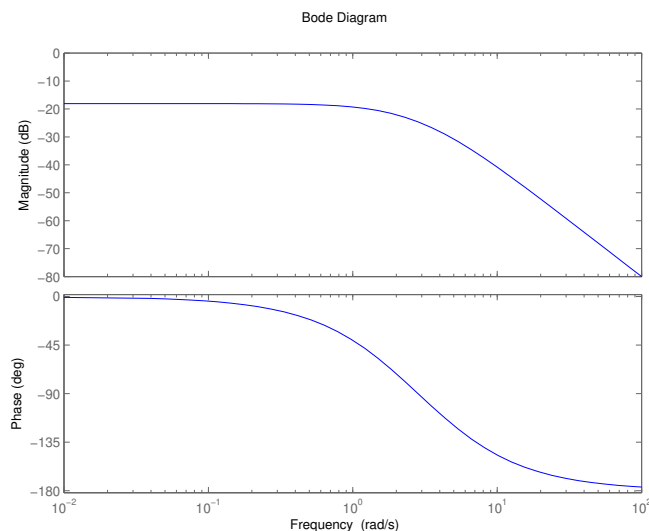
$$\text{a) } G(s) = \frac{1}{(s+2)(s+4)} \quad \text{b) } G(s) = \frac{1}{(s+1)(s^2+s+1)}$$

*Instructions:* To draw the Nyquist plot, use the Bode plots of  $G(s)$ . Explain all steps in arriving at the Nyquist plot. Start with hand-sketched Bode plots. You can then generate more accurate Bode plots with MATLAB to get exact numerical values if necessary. *The use of MATLAB for drawing the Nyquist plot is not allowed* (except to check your work at the end). It is also recommended that you check your results with Routh stability criterion.

*Solution:*

(a)

$$G(s) = \frac{1}{(s+2)(s+4)}$$



$$\omega \rightarrow 0 \implies |G(j\omega)| = \frac{1}{8}, \quad \angle G(j\omega) = 0^\circ$$

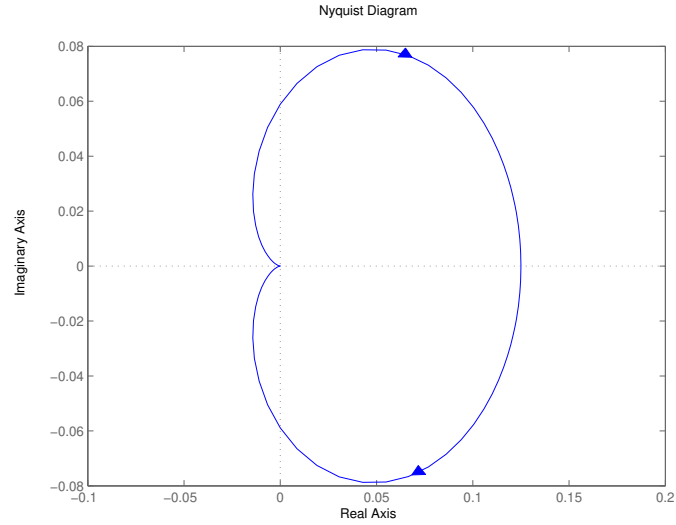
$$\omega = \sqrt{8} \implies |G(j\omega)| = \frac{1}{6\sqrt{8}}, \quad \angle G(j\omega) = -90^\circ$$

$$\omega \rightarrow \infty \implies |G(j\omega)| \rightarrow 0, \quad \angle G(j\omega) = -180^\circ$$

$P$  : # RHP poles = 0

$Z$  : # of closed loop RHP poles

$N$  : # of encirclements of  $-1/K$



$$Z = N + P \implies Z = N$$

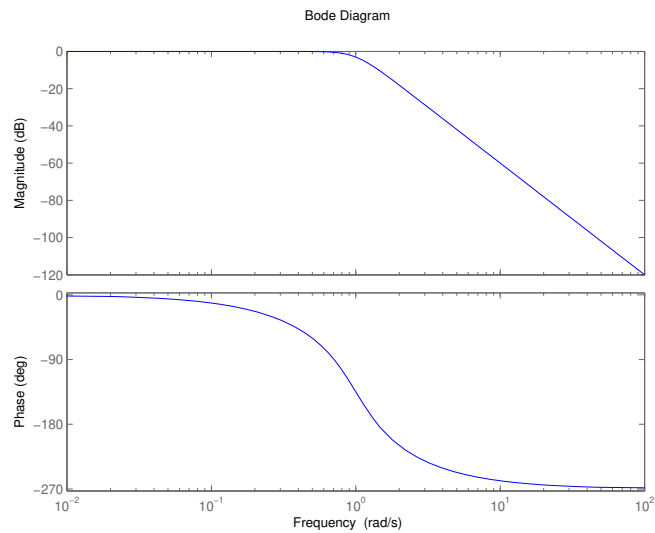
if  $K > 0 \implies -\frac{1}{K} < 0, N = 0$  according to Nyquist plot

if  $-8 < K \leq 0 \implies -\frac{1}{K} > \frac{1}{8}, N = 0$

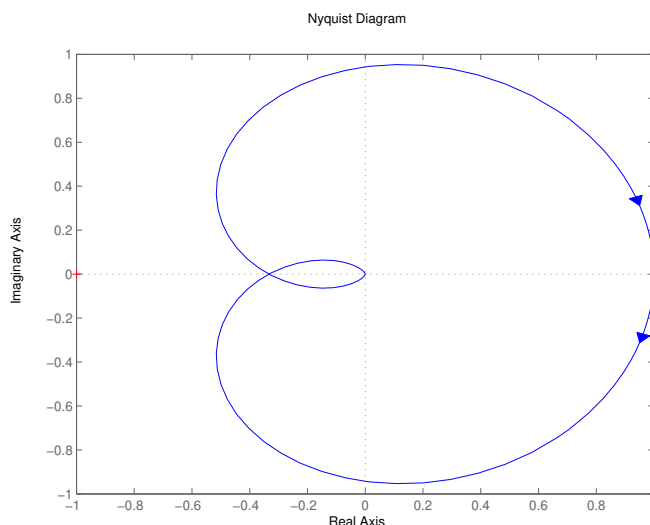
if  $K < -8 \implies 0 < -\frac{1}{K} < \frac{1}{8} \implies N = 1 \implies Z = 1 \implies$  unstable closed-loop  
 $\implies$  for stability we need  $K \geq -8$

(b)

$$G(s) = \frac{1}{(s+1)(s^2+s+1)}$$



$$\begin{aligned} \omega \rightarrow 0 &\implies |G(j\omega)| = 1, \angle G(j\omega) = 0^\circ \\ \omega = \frac{1}{\sqrt{2}} &\implies |G(j\omega)| = 0.95, \angle G(j\omega) = -90^\circ \\ \omega = \sqrt{2} &\implies |G(j\omega)| = \frac{1}{3}, \angle G(j\omega) = -180^\circ \\ \omega \rightarrow \infty &\implies |G(j\omega)| \rightarrow 0, \angle G(j\omega) = -270^\circ \end{aligned}$$



$P = 0$ . We can see that if  $-\frac{1}{K} \in (-\frac{1}{3}, 1)$  then  $N > 0$  and closed-loop system would be unstable for stability we need  $-1 < K < 3$  so that  $N = 0$

2. Consider the system

$$\dot{x} = \begin{pmatrix} 0 & -1 & 2/3 \\ -1 & -2 & 1 \\ 0 & -3 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} u, \quad y = x_2$$

a) Write down the open-loop characteristic equation. (This involves computing a  $3 \times 3$  determinant, which you can do either by hand or in MATLAB using a symbolic variable  $s$ .) Are all open-loop poles in the LHP?

b) Using the formula given in class, compute the transfer function of this system. (Use the general formula, do *not* take Laplace transform of individual differential equations. Look up the procedure for inverting a matrix by hand, or use the MATLAB command `inv`.)

c) Find another state-space realization of the same transfer function, in controller canonical form

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u, \quad y = (b_3 \ b_2 \ b_1) x$$

Hint: you should see that, similarly to the  $2 \times 2$  case discussed in class, there is a simple relationship between the entries in the above matrices and the coefficients in the transfer function.

*Solution:*

$$\dot{x} = \begin{pmatrix} 0 & -1 & \frac{2}{3} \\ -1 & -2 & 1 \\ 0 & -3 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} u, \quad y = x_2$$

(a) Open-loop characteristic equation

$$\det(sI - A) = \det \left\{ \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} 0 & -1 & \frac{2}{3} \\ -1 & -2 & 1 \\ 0 & -3 & 1 \end{pmatrix} \right\}$$

$$= s^3 + s^2 - 1 \quad (\text{all coefficients are NOT positive})$$

open-loop poles  $\equiv$  roots of O-L characteristic equation  
 $\equiv \{0.75, -0.88 \pm 0.75j\} \implies$  O-L unstable

(b)

$$G(s) = C(sI - A)^{-1}B = (0 \quad 1 \quad 0) (sI - A)^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$(sI - A)^{-1} = \begin{pmatrix} s & 1 & -2/3 \\ 1 & s+2 & -1 \\ 0 & 3 & s-1 \end{pmatrix}^{-1} = \frac{1}{s^3 + s^2 - 1} \begin{pmatrix} s^2 + s + 1 & -s - 1 & \frac{1+2s}{3} \\ -s + 1 & s^2 - s & s - 2/3 \\ 3 & -3s & s^2 + 2s - 1 \end{pmatrix}$$

$$\implies G(s) = \frac{2s^2 - 1}{s^3 + s^2 - 1} = \frac{b_1s^2 + b_2s + b_3}{s^3 + a_1s^2 + a_2s + a_3}$$

(c) Controller canonical form:  $\{b_1 = 2, b_2 = 0, b_3 = -1, a_1 = 1, a_2 = 0, a_3 = -1\}$

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u, \quad y = (-1 \quad 0 \quad 2) x$$

3. Determine (from the controllability matrix) whether or not the following systems are controllable.

a)  $\dot{x} = \begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$       b)  $\dot{x} = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 3 \\ -1 & -1 & -2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u$

You can use MATLAB to perform matrix multiplication, but you should know how to do it by hand.

*Solution:*

(a)

$$\dot{x} = \begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

Controllability matrix:  $\mathcal{C}(A, B) = (B : AB)$

$$\mathcal{C}(A, B) = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}, \quad \det(\mathcal{C}(A, B)) = -2 \neq 0$$

$$\implies \mathcal{C}(A, B) \text{ is full-rank, system is controllable}$$

(b)

$$\dot{x} = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 3 \\ -1 & -1 & -2 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u$$

$$\mathcal{C}(A, B) = (B : AB : A^2B) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & -2 \\ 0 & -1 & 1 \end{pmatrix}, \quad \text{rank}(\mathcal{C}(A, B)) = 2 < 3 (\text{or } \det(\mathcal{C}(A, B)) = 0)$$

System is NOT controllable

4. Imagine two cars driving on the same road in the same direction one behind the other and trying to go at the same speed. This situation can be described by the linear system

$$\begin{aligned}\dot{v}_1 &= -(v_1 - v_2) + f_1 \\ \dot{v}_2 &= -(v_2 - v_1) + f_2\end{aligned}$$

where, for  $i = 1, 2$ ,  $v_i$  is the velocity of car  $i$  and  $f_i$  is an external force (wind, road conditions, etc.) acting on it. The meaning of the above equations is that each car accelerates/decelerates depending on whether it is going slower/faster than the other. Now, suppose that we want to rewrite the above system in the following new coordinates:  $\bar{v}_1 := v_1$  (velocity of car 1) and  $\bar{v}_2 := v_2 - v_1$  (relative velocity of the two cars).

a) Write down the differential equations for  $\dot{\bar{v}}_1, \dot{\bar{v}}_2$  in terms of  $\bar{v}_1, \bar{v}_2$  and  $f_1, f_2$ .

b) Write down the original system in state-space form  $\dot{v} = Av + Bf$ , the new system in state-space form  $\dot{\bar{v}} = \bar{A}\bar{v} + \bar{B}f$ , the coordinate transformation matrix  $T$  from  $(v_1, v_2)$  to  $(\bar{v}_1, \bar{v}_2)$ , and verify that the formulas given in class (relating  $\bar{A}$  with  $A$  and  $\bar{B}$  with  $B$  via  $T$ ) hold. (Hint: since we have two inputs,  $B$  is a matrix, not a vector.)

*Solution:*

$$\begin{aligned}\dot{v}_1 &= -(v_1 - v_2) + f_1 = \bar{v}_2 + f_1 \\ \dot{v}_2 &= -(v_2 - v_1) + f_2 = -\bar{v}_2 + f_2\end{aligned}$$

$$\implies \underbrace{\begin{pmatrix} \dot{v}_1 \\ \dot{v}_2 \end{pmatrix}}_{\dot{v}} = \underbrace{\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}}_A \underbrace{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}_v + \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_B \underbrace{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}}_u$$

(a)

$$\begin{aligned}\dot{\bar{v}}_1 &= \dot{v}_1 = +\bar{v}_2 + f_1 \\ \dot{\bar{v}}_2 &= \dot{v}_2 - \dot{v}_1 = -\bar{v}_2 + f_2 - (\bar{v}_2 + f_1) = -2\bar{v}_2 - f_1 + f_2\end{aligned}$$

$$\implies \begin{cases} \dot{\bar{v}}_1 = \bar{v}_2 + f_1 \\ \dot{\bar{v}}_2 = -2\bar{v}_2 + f_2 - f_1 \end{cases}$$

(b)

$$\begin{aligned}\bar{v}_1 &= v_1 \\ \bar{v}_2 &= v_2 - v_1\end{aligned} \implies \underbrace{\begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix}}_{\bar{v}} = \underbrace{\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}}_T \underbrace{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}_v$$

$$\begin{aligned}\dot{\bar{v}} &= T\dot{v} = TAv + TBu \\ \dot{\bar{v}} &= TAT^{-1}\bar{v} + TBu \\ &= \underbrace{\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}}_T \underbrace{\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}}_A \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}_{T^{-1}} \bar{v} + \underbrace{\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}}_T \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_B \underbrace{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}}_u \\ &= \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix}}_{\bar{A}} \bar{v} + \underbrace{\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}}_{\bar{B}} u \\ \dot{\bar{v}}_1 &= \bar{v}_2 + f_1 \\ \dot{\bar{v}}_2 &= -2\bar{v}_2 - f_1 + f_2\end{aligned}$$