Explaining the Routh–Hurwitz Criterion

A Tutorial Presentation

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Routh's treatise [1] was a landmark in the analysis of the stability of dynamic systems and became a core foundation of control theory. The remarkable simplicity of the result was in stark contrast to the challenge of the proof. Many researchers devoted much effort to extend the result to singular cases, with some of the earlier techniques shown to be inadequate [2]. Together with the extensions to singular cases, shorter proofs were also proposed. The proof of [3] is noteworthy, which followed the root locus arguments of [4]. A key feature of the proof is a continuity argument used in an earlier derivation [5]. In [6], the more conventional approach using Cauchy's principle of the argument is followed. A relatively simple proof is proposed, considering the extension to complex polynomials and singular cases.

Control textbooks describe the Routh–Hurwitz criterion but do not explain how the result is obtained. Consequently, the procedure remains mysterious to many students and their teachers. "Summary" gives a brief overview of the results achieved in this article. The main contributions are to show that the interpretation of the Routh array is straightforward and that two proofs of the criterion can be completed shortly. The first proof is based on [3], and the second is inspired from [6], using the Nyquist criterion instead of Cauchy's principle. The second proof is also similar to the one in [7]. Small changes are made to the proofs to remove some technical steps and further simplify them. The derivations require only standard knowledge available from textbooks on feedback systems.

Given the computing power available today, the Routh–Hurwitz criterion has lost some of its importance, but it remains valuable in practical problems. The procedure makes it possible to obtain analytic stability conditions for specific applications involving multiple plant and controller parameters (see "Applications of the Routh–Hurwitz Criterion"). Overall, the Routh–Hurwitz criterion remains a remarkable result of historical significance.

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Summary

The Routh–Hurwitz criterion is a mathematical tool used to determine whether all of the roots of a polynomial have negative real parts. The algorithm makes it possible to determine whether a closed-loop system is stable, including the conditions needed on plant and controller parameters to achieve stability. The procedure of the Routh–Hurwitz criterion is relatively simple. However, the proof of the result has been elusive to students and their teachers. This article shows that an explanation of the Routh–Hurwitz criterion can be presented shortly at the level of an introductory control course.

THE ROUTH–HURWITZ CRITERION

Consider a polynomial

$$p(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0. \tag{1}$$

The first two rows of the *Routh array* are obtained by copying the coefficients of $p(s)$ using the pattern

$$\begin{array}{c|cccccc} s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots & \cdots \\ s^{n-2} & x_1 & x_2 & x_3 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}.$$

When a_0 is reached in one of the first two rows, blanks are left in the remaining slots, which are equivalent to zeros. The first two rows are labeled s^n and s^{n-1}, respectively. The third row is labeled s^{n-2} and has elements

$$x_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}},\; x_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}, \cdots. \tag{2}$$

The computation is repeated for subsequent rows until the row labeled s^0 is reached. The case is called *regular* if no coefficient of the first column (also called a leading coefficient) is zero. Otherwise, the case is called *singular*, and the algorithm stops prematurely.

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Applications of the Routh–Hurwitz Criterion

Although the roots of polynomials are easily computed numerically, the Routh–Hurwitz criterion remains useful to determine how stability is affected by multiple plant and controller parameters. In [S1], a bound is derived for the gain of a dc–dc buck converter as a function of five system parameters. The minimum input voltage required for the stable operation of a type 3 phase-locked loop is obtained in [S2], whereas a condition relates the four circuit parameters of a constant-power load damper circuit in [S3]. Sometimes, the objective is to achieve *instability*, that is, in the design of an oscillator in [S4]. For the control of a remotely piloted aircraft [S5], the Routh–Hurwitz criterion gives a condition to be satisfied by the load parameters, so that stability is guaranteed. The condition is a function of the mass and inertia of the helicopter, aerodynamic parameters, and controller parameters. A set of inequalities is obtained in [S6] to ensure that a fixed-structure/fixed-order controller using Groebner bases is stabilizing.

Less conventional applications can be found, that is, the synchronization of fractional-order chaotic systems, with applications to cryptography [S7]. The authors of [S8] address the stability of the dynamics of HIV infection and drug therapy, and the paper is representative of a class of studies where the Routh–Hurwitz criterion is used to evaluate the stability of a biological model. Similarly, the stability of genetic circuits is the focus of [S9]. The extension of the stability test to systems with complex parameters is considered in [S10], but the paper uses the Hurwitz determinants instead of the Routh array. The sixth-order model of a self-excited induction generator is transformed into an equivalent third-order system with complex coefficients, and analytic conditions are deduced for the instability of the zero equilibrium (a necessary condition for generation). In [S11], a simple condition is found to ensure the stability of a two-input, two-output proportional-integral-control law applied to a doubly fed induction generator.

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If the case is regular, the Routh–Hurwitz criterion states that the number of right half-plane (RHP) roots of the polynomial $p(s)$ is equal to the number of sign changes in the first column of the array. The RHP [or left half-plane (LHP)] is taken to be the part of the plane such that $\mathrm{Re}(s) > 0$ [or $\mathrm{Re}(s) < 0$]. There can be no root on the imaginary axis [such that $\mathrm{Re}(s) = 0$] in the regular case. Conversely, if the roots are in the LHP, the case must be regular. Therefore, the Routh–Hurwitz criterion implies that the roots of $p(s)$ are in the LHP if and only if all of the elements of the first column are nonzero and have the same signs.

EXPLANATION OF THE ROUTH ARRAY

The first two rows of the array contain the coefficients of the polynomials

$$p_1(s) = a_n s^n + a_{n-2} s^{n-2} + \cdots, \tag{3}$$

$$p_2(s) = a_{n-1} s^{n-1} + a_{n-3} s^{n-3} + \cdots. \tag{4}$$

The elements that are zero by construction are omitted from the array. One of the polynomials $p_1(s)$ and $p_2(s)$ is even (that is, only has even powers of s, including s^0), and the other polynomial is odd (only has odd powers of s). A polynomial $p_3(s)$ is defined that is the remainder of the division of polynomial $p_1(s)$ by $p_2(s)$, so that

$$p_1(s) = q_1(s) p_2(s) + p_3(s), \tag{5}$$

where $q_1(s) = a_n s / a_{n-1}$ is the quotient. The third row of the array contains the coefficients of the remainder:

$$p_3(s) = (a_{n-2} - a_{n-3} a_n / a_{n-1}) s^{n-2} + (a_{n-4} - a_{n-5} a_n / a_{n-1}) s^{n-4} + \cdots. \tag{6}$$

Repeating the procedure, polynomials $p_k(s)$ are constructed so that

$$p_{k+2}(s) = p_k(s) - q_k(s) p_{k+1}(s) \quad \text{for } k = 1, \ldots, n-1. \tag{7}$$

The polynomials $p_k(s)$ are of the form

$$p_k(s) = c_k s^{n-k+1} + \cdots. \quad (8)$$

Note that c_k is the leading coefficient of row k, with $c_1 = a_n$ and $c_2 = a_{n-1}$. The quotient polynomials are given by

$$q_k(s) = \frac{c_k}{c_{k+1}} s \quad \text{for } k = 1, \ldots, n-1. \quad (9)$$

The polynomials $p_k(s)$ alternate as even and odd polynomials of decreasing order. The Routh array contains the coefficients of these polynomials, omitting the coefficients that are always equal to zero due to the even/odd property. The labels on the left of the array give the highest power of s of the polynomials. If no c_k is equal to zero, the last two polynomials of the sequence are $p_n(s) = c_n s$ and $p_{n+1}(s) = c_{n+1}$.

Together with the polynomials $p_k(s)$, the procedure also produces a sequence of polynomials $p_k(s) + p_{k+1}(s)$, starting from the original polynomial $p(s) = p_1(s) + p_2(s)$. The Routh–Hurwitz criterion originates from a key property that applies to these polynomials at every step of the procedure.

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Key Property

Assuming that $c_1, \ldots, c_{k+1} \neq 0$, the number of roots of $p_k(s) + p_{k+1}(s)$ with $\mathrm{Re}(s) < 0$ [or $\mathrm{Re}(s) > 0$] is equal to the number of roots of $(1 + q_k(s))(p_{k+1}(s) + p_{k+2}(s))$ with $\mathrm{Re}(s) < 0$ [or $\mathrm{Re}(s) > 0$]. The roots with $\mathrm{Re}(s) = 0$ are identical in both polynomials, including their multiplicity.

Note that the last polynomial in the sequence is $p_n(s) + p_{n+1}(s) = c_n s + c_{n+1}$. Given that $1 + q_k(s) = (c_k s + c_{k+1})/c_{k+1}$, the Routh–Hurwitz criterion follows from the key property in a straightforward manner. One can also conclude the following:

- A case where $p(s)$ has imaginary roots must be singular. Indeed, $1 + q_k(s)$ and $c_n s + c_{n+1}$ can only have real roots, so that the procedure must stop before the last step if there are imaginary roots.
- A case with $c_{k+1} = 0$ for some k has roots with $\mathrm{Re}(s) \geq 0$. Indeed, $c_{k+1} = 0$ if and only if the second coefficient of $p_k(s) + p_{k+1}(s)$ is zero. The second coefficient is the sum of the roots of $p_k(s) + p_{k+1}(s)$, which implies that some roots must be on the imaginary axis or in the RHP. The original polynomial must have at least the same number of roots with $\mathrm{Re}(s) \geq 0$.
- Conversely, a case where $p(s)$ has all roots with $\mathrm{Re}(s) < 0$ must be regular.

FIRST PROOF OF THE KEY PROPERTY USING CONTINUITY

The proof relies on the even/odd nature of the polynomials and properties that are straightforward to prove. An even polynomial $p_e(s)$ is such that $p_e(j\omega)$ is purely real. With $p_e(s) = p_e(-s)$, its roots must be pairs of imaginary roots ($s = \pm jb$), pairs of real roots ($s = \pm a$), or quadruples of complex roots ($s = \pm a \pm jb$). An odd polynomial $p_o(s)$ is such that $p_o(j\omega)$ is purely imaginary and $p_o(s) = s\, p_e(s)$, where $p_e(s)$ is an even polynomial. Its roots must include a root at $s = 0$, plus the same types of roots as an even polynomial. The sum of two even/odd polynomials is even/odd. The product of two even or two odd polynomials is even, and the product of an even polynomial with an odd polynomial is odd.

The proof presented here is mostly the same as the one found in [3], with a small simplification obtained by considering a different polynomial in the analysis. The polynomial is

$$d_{k,g}(s) = p_k(s) + p_{k+1}(s) + g\, q_k(s)\, p_{k+2}(s) \quad (10)$$

$$= p_{k+2}(s) + q_k(s) p_{k+1}(s) + p_{k+1}(s) + g\, q_k(s)\, p_{k+2}(s), \quad (11)$$

where $g \in [0, 1]$. For $g = 0$, $d_{k,0}(s) = p_k(s) + p_{k+1}(s)$, whereas for $g = 1$,

$$d_{k,1}(s) = (1 + q_k(s))(p_{k+1}(s) + p_{k+2}(s)). \quad (12)$$

The polynomial $d_{k,g}(s)$ in (10) is the sum of $p_k(s)$ and two polynomials of lower degree. Therefore, $d_{k,g}(s)$ has degree $n - k + 1$ for all $g \in [0, 1]$, and continuous branches connect the roots of $d_{k,0}(s)$ to the roots of $d_{k,1}(s)$.

Next, note that a root of $d_{k,g}(s)$ belongs to the imaginary axis if and only if, for some ω_0,

$$p_{k+2}(j\omega_0) + q_k(j\omega_0) p_{k+1}(j\omega_0) + p_{k+1}(j\omega_0) + g\, q_k(j\omega_0) p_{k+2}(j\omega_0) = 0. \quad (13)$$

Due to the even/odd alternation of the polynomials $p_k(s)$ and with $q_k(s)$ being an odd polynomial, the equation can be split into real and imaginary parts as

$$p_{k+2}(j\omega_0) + q_k(j\omega_0) p_{k+1}(j\omega_0) = 0, \quad (14)$$

$$p_{k+1}(j\omega_0) + g\, q_k(j\omega_0) p_{k+2}(j\omega_0) = 0. \quad (15)$$

It follows that

$$(1 - g\, q_k^2(j\omega_0))\, p_{k+2}(j\omega_0) = 0. \quad (16)$$

In (16), $1 - g\, q_k^2(j\omega_0) = 1 + g(c_k \omega_0 / c_{k+1})^2 \geq 1$, $p_{k+2}(j\omega_0) = 0$, and $p_{k+1}(j\omega_0) = 0$ as well. This result is true for all $g \in [0, 1]$, so that any root of $d_{k,g}(s)$ on the imaginary axis for some

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g is a root of $p_k(s)+p_{k+1}(s)$, a root of $p_{k+1}(s)+p_{k+2}(s)$, and a root of $d_{k,g}(s)$ for *all* g. Imaginary roots remain at their location, and no root of $d_{k,g}(s)$ can move from the RHP or the LHP to the imaginary axis. Therefore, no root can also move from the RHP to the LHP and vice versa. The key property follows.

SECOND PROOF OF THE KEY PROPERTY USING THE NYQUIST CRITERION

The key property can also be proved using the Nyquist criterion, and we assume that $p_k(s)+p_{k+1}(s)$ and $p_{k+1}(s)+p_{k+2}(s)$ have no roots on the imaginary axis to keep the proof simple. Consider the open-loop transfer function

$$G_k(s)=\frac{-q_k(s)\,p_{k+2}(s)}{(1+q_k(s))\,(p_{k+1}(s)+p_{k+2}(s))}. \tag{17}$$

The poles of this transfer function are the roots of

$$p_{\mathrm{ol}}(s)=(1+q_k(s))\,(p_{k+1}(s)+p_{k+2}(s)), \tag{18}$$

whereas the poles of the closed-loop transfer function $G_k(s)/(1+G_k(s))$ are the roots of

$$p_{\mathrm{cl}}(s)=(1+q_k(s))\,(p_{k+1}(s)+p_{k+2}(s))-q_k(s)\,p_{k+2}(s) \tag{19}$$

$$=p_k(s)+p_{k+1}(s). \tag{20}$$

The Nyquist criterion specifies that the number of RHP roots of $p_{\mathrm{cl}}(s)$ is equal to the number of RHP roots of $p_{\mathrm{ol}}(s)$ plus the number of clockwise encirclements of $(-1,0)$ by the curve $G_k(s)$ computed along the Nyquist contour. Because $G_k(s)$ has more poles than zeros, $\lim_{\omega\to\infty}G_k(j\omega)=\lim_{\omega\to-\infty}G_k(j\omega)=0$. Also, $G_k(0)=0$ because $q_k(s)$ has a zero at $s=0$. With no pole on the imaginary axis, the Nyquist curve is a bounded and closed curve that reaches the origin for $\omega=0$ and $\omega\to\pm\infty$. Note that, for $c_{k+1}\neq 0$,

$$\left|\frac{q_k(j\omega)}{1+q_k(j\omega)}\right|=\left|\frac{jc_k\omega}{c_{k+1}+jc_k\omega}\right|<1 \quad \text{for all } \omega. \tag{21}$$

Similarly, $p_{k+1}(j\omega)$ is real and $p_{k+2}(j\omega)$ is imaginary, or vice versa, so that

$$\left|\frac{p_{k+2}(j\omega)}{p_{k+1}(j\omega)+p_{k+2}(j\omega)}\right|\leq 1 \quad \text{for all } \omega. \tag{22}$$

It follows that $|G_k(j\omega)|<1$ for all ω, including as $\omega\to\pm\infty$. As a result, there can be no encirclements of $(-1,0)$ by the Nyquist curve and the key property follows.

SINGULAR CASES

The regular procedure stops when the leading coefficient $c_{k+1}=0$. Two singular cases can be defined.

- *Singular case 1*: The leading coefficient is zero, but the row is not identically zero. Polynomial division could proceed but would produce an odd polynomial $q_k(s)$ of degree three (or higher if the next coefficient is also zero). The sum of the roots of $1+q_k(s)$ would be equal to zero, so that some roots would not be in the LHP.
- *Singular case 2*: The row of the Routh array is identically zero, so that $p_{k+2}(s)=0$ and $p_{k+1}(s)+p_{k+2}(s)=p_{k+1}(s)$ (which is either even or odd). Some roots of $p_{k+1}(s)+p_{k+2}(s)$ would not be in the LHP.

The two cases confirm that the polynomial $p(s)$ cannot have all roots with $\mathrm{Re}(s)<0$ if some leading coefficient of the array is equal to zero. To continue counting the roots in the singular case, an alternate procedure is needed. In the most recent work, the preferred approach has replaced $p_{k+1}(s)+p_{k+2}(s)$ with a polynomial to which the regular procedure can be applied and the root locations can be related. The author of [3] gives an approach for singular cases based on [8] and provides a short Matlab code to count the roots in the RHP and LHP and on the imaginary axis. However, the main justification for counting the roots in the singular case is to determine whether a system is marginally stable. Therefore, one needs to know whether any root on the imaginary axis is repeated. The authors of [9] and [10] propose Routh-like procedures for singular cases to determine whether any imaginary root is repeated. Still, the usefulness of procedures for singular cases is limited from a practical perspective, because a system is known to be bounded-input, bounded-output unstable as soon as a zero-leading coefficient is encountered in the Routh array.

INVARIANT ROOTS

The key property implies that imaginary roots remain invariant at every step of the procedure. Interestingly, other roots are invariant as well. In [11], it was observed that the roots of the polynomial $p_{k+1}(s)$ in singular case 2 must appear in the original polynomial $p(s)$. This property follows from the recursion

$$p_k(s)=q_k(s)\,p_{k+1}(s)+p_{k+2}(s). \tag{23}$$

With $p_{k+2}(s)=0$, $p_k(s)$ must be a multiple of $p_{k+1}(s)$. Similarly, $p_{k-1}(s)$ must be a multiple of $p_{k+1}(s)$ as well as every $p_j(s)$ for $j<k$. It follows that $p(s)$ must be a multiple of the last nonzero polynomial $p_{k+1}(s)$.

Conversely, start from a polynomial $p(s) = p_a(s)p_m(s)$, where $p_m(s)$ is an even polynomial. Letting $p_a(s) = p_e(s) + p_o(s)$ [where $p_e(s)$ is even and $p_o(s)$ is odd], $p(s)$ is the sum of the even polynomial $p_e(s)p_m(s)$ and the odd polynomial $p_o(s)p_m(s) \cdot p_1(s)$, and $p_2(s)$ is equal to these two polynomials and is therefore a multiple of $p_m(s)$. The same result is true if $p_m(s)$ is an odd polynomial. From (23), every $p_k(s)$ is a multiple of $p_m(s)$ until the procedure stops.

The conclusion is that, if $p(s)$ is the multiple of an even or odd polynomial, every polynomial $p_k(s) + p_{k+1}(s)$ is a multiple of that polynomial. As a result, not only are purely imaginary roots invariant in the procedure but also *any* pair of roots symmetric with respect to the imaginary axis. The presence of such roots in the polynomial $p(s)$ implies that the case must be singular.

The proof relies on the even/odd nature of the polynomials and properties that are straightforward to prove.

EXAMPLES

Example 1: Using the Routh–Hurwitz Criterion to Find Stability Conditions

Consider the control system of Figure 1. The plant is an electric motor with an inner torque control loop, resulting in

$$\theta = \frac{1}{Js^2}\tau_{\mathrm{COM}}, \tag{24}$$

where θ is the angular position of the motor (in rad), J is the inertia of the motor and load (in kg-m^2), and τ_{COM} is the torque command (in N-m). The controller is a proportional-integral-derivative (PID) control law

$$\tau_{\mathrm{COM}} = \left(k_P + \frac{k_I}{s}\right)(\theta_{\mathrm{REF}} - \theta) - k_D\frac{a_F s}{s + a_F}\theta, \tag{25}$$

where θ_{REF} is the reference input for the position and k_P, k_I, and k_D are the PID gains. The derivative term is filtered by a first-order system with a pole at $s = -a_F$ to reduce the high-frequency noise originating from the differentiation of the position measurement. The derivative action is not applied to the reference input to avoid large transients when step inputs are applied. The objective is to find conditions on the PID gains so that the closed-loop system is stable. J and a_F are positive parameters.

The closed-loop polynomial is

$$p(s) = Js^4 + Ja_Fs^3 + (k_P + k_Da_F)s^2 + (k_Pa_F + k_I)s + k_Ia_F, \tag{26}$$

so that the Routh array is given by

$$\begin{array}{c|ccc} s^4 & J & k_P + k_D a_F & k_I a_F \\ s^3 & Ja_F & k_P a_F + k_I & \\ s^2 & x_1 & k_I a_F & \\ s^1 & y_1 & & \\ s^0 & k_I a_F & & \end{array},$$

where

$$x_1 = k_D a_F - k_I/a_F,\ \ y_1 = k_P a_F + k_I - \frac{Jk_I a_F^2}{x_1}. \tag{27}$$

It follows that the conditions that the PID gains must satisfy for stability are

$$k_I > 0,\ \ k_D > \frac{k_I}{a_F^2},\ \ k_P > \frac{Jk_I a_F^2}{k_D a_F^2 - k_I} - \frac{k_I}{a_F}. \tag{28}$$

Example 2: Root Locus in a Regular Case

Consider the polynomial $p(s) = s^6 + 4s^5 + 8s^4 + 6s^3 + s^2 + 10s + 50$, with the Routh array

$$\begin{array}{c|cccc} s^6 & 1 & 8 & 1 & 50 \\ s^5 & 4 & 6 & 10 & \\ s^4 & 6.5 & -1.5 & 50 & \\ s^3 & 6.92 & -20.77 & & \\ s^2 & 18 & 50 & & \\ s^1 & -40 & & & \\ s^0 & 50 & & & \end{array}.$$

Figure 2 shows the root locus obtained through the procedure of the Routh–Hurwitz criterion. The locus is a sequence of root loci truncated to $g \in [0, 1]$ rather than a single conventional root locus with $g \in [0, \infty)$. The locations of the roots at each step are marked by red dots. The roots of $p_1(s) + p_2(s)$ are marked with the green label 1. For $k > 1$, the roots of $(1 + q_k(s))(p_{k+1}(s) + p_{k+2}(s))$ are identified by the number $k + 1$, with the label for the root of $1 + q_k(s)$ placed in a box. Such a root marks the end of a branch. The procedure is repeated at every step with a decreasing number

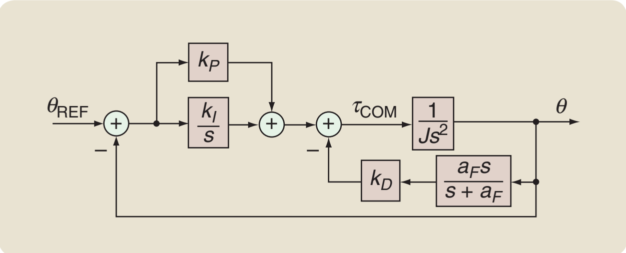

FIGURE 1 The proportional-integral-derivative control scheme for an electric motor. θ_{REF} is the reference position, τ_{COM} is the torque command, and θ is the angular position of the motor. A first-order filter is integrated with the derivative term.

of roots. All roots end their journey on the real axis and on the same side of the imaginary axis as the side from which they started.

Example 3: Root Locus in a Singular Case With Imaginary Roots

Consider the polynomial $p(s) = s^6 + 2s^5 + 3s^4 + 26s^3 + 26s^2 + 72s + 720$. The polynomial has a pair of imaginary roots, so that the Routh array stops before the end:

$$\begin{array}{c|cccc} s^6 & 1 & 3 & 26 & 720 \\ s^5 & 2 & 26 & 72 & \\ s^4 & -10 & -10 & 720 & \\ s^3 & 24 & 216 & & \\ s^2 & 80 & 720 & & \\ s^1 & 0 & & & \end{array}.$$

The example corresponds to singular case 2, with the row s^1 equal to zero. The root locus is shown on Figure 3. Note that the imaginary roots do not move throughout the procedure. The other roots reach the real axis, and the algorithm stops when the two imaginary roots remain alone. The roots of $p_5(s) + p_6(s) = 80s^2 + 720$ are the same as the original imaginary roots at $s = \pm j3$.

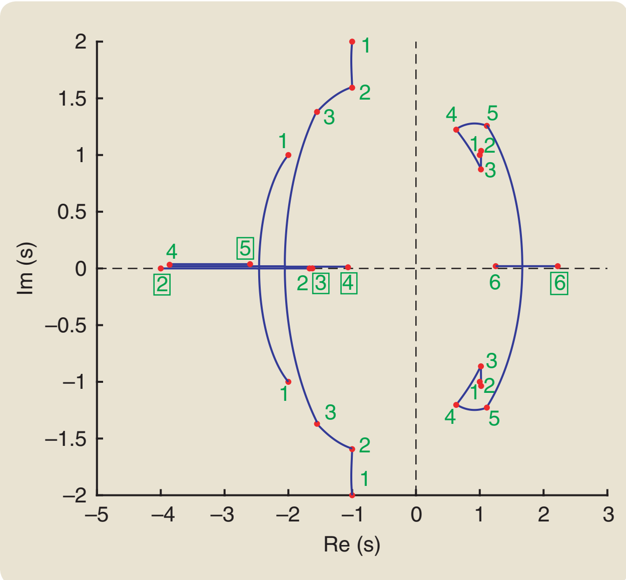

FIGURE 2 The root locus plot for a regular case. The roots move at every step but remain on the same side of the imaginary axis. Roots in a box are roots of $1 + q_k(s)$ and mark the end of a branch.

Example 4: Root Locus in a Singular Case Without Imaginary Roots

Consider the polynomial $p(s) = s^5 + 2s^4 + 3s^3 + 2s^2 + 3s + 2$, with the Routh array

$$\begin{array}{c|ccc} s^5 & 1 & 3 & 3 \\ s^4 & 2 & 2 & 2 \\ s^3 & 2 & 2 & 0 \\ s^2 & 0 & 2 & 0 \end{array}.$$

The procedure ends prematurely after two steps, even though there are no imaginary roots. The example corresponds to singular case 1, with the leading coefficient of row s^2 equal to zero. The root locus is shown on Figure 4. The last polynomial is $p_3(s) + p_4(s) = 2s^3 + 2s + 2$ and has roots at $0.3412 \pm 1.1615j$ and -0.6823. The sum of the roots is equal to zero. These roots are marked with the label 3 (without the box) on Figure 4.

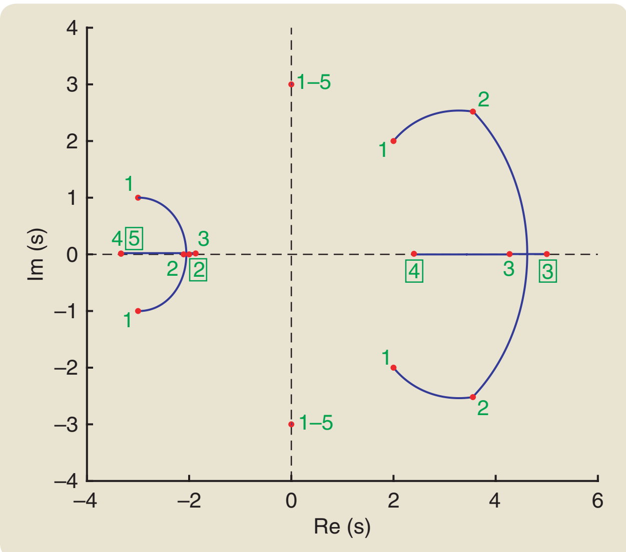

FIGURE 3 The root locus plot for a singular case with imaginary roots. The roots with nonzero real parts remain on the same side of the imaginary axis. The imaginary roots remain in the same location, eventually causing the procedure to stop with a zero leading coefficient in the Routh array.

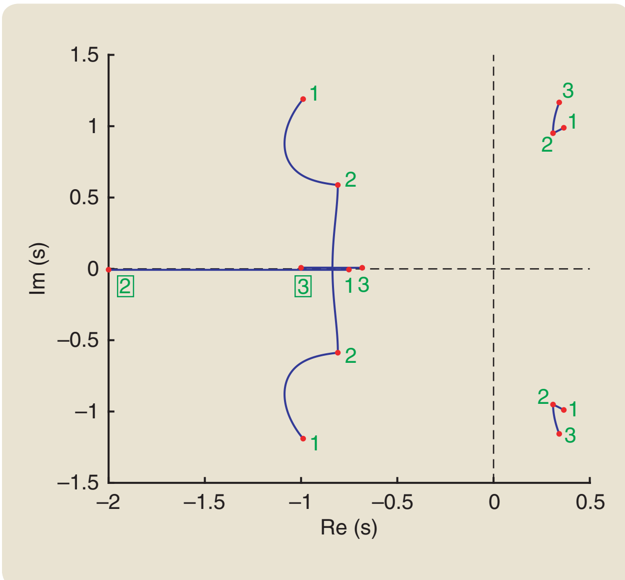

FIGURE 4 The root locus plot for a singular case without imaginary roots. The procedure stops because the sum of the three roots labeled 3 (without the box) is equal to zero, causing a leading coefficient of the Routh array to be equal to zero.

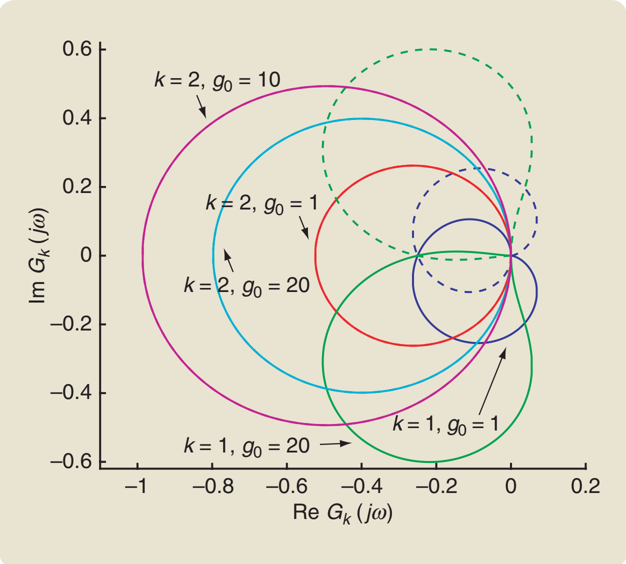

FIGURE 5 The Nyquist plots associated with the second proof. All of the Nyquist curves fit strictly inside a circle of magnitude one, implying that the number of right and left half-plane roots are the same in the two polynomials.

Example 5: Nyquist Diagram

Consider the polynomial $p(s) = s^3 + 3s^2 + 3s + (1 + g_0)$, with the Routh array

$$\begin{array}{c|cc} s^3 & 1 & 3 \\ s^2 & 3 & (1+g_0) \\ s^1 & (8-g_0)/3 & \\ s^0 & 1+g_0 & \end{array}.$$

The Routh–Hurwitz criterion implies that no roots of $p(s)$ lie in the RHP if $-1 < g_0 < 8$. For $g_0 > 8$, there are two sign changes and therefore two roots in the RHP. Figure 5 shows the Nyquist plots of $G_k(s)$ for $k = 1$ and $k = 2$ and for $g_0 = 1$ and $g_0 = 20$. A third curve shows the Nyquist plot for $k = 2$ and $g_0 = 10$ (the $k = 1$, $g_0 = 10$ curve is omitted to avoid overloading the plot). The positive and negative frequency curves for $k = 2$ overlap exactly in this example.

There are no encirclements of (–1, 0) by any curve because $|G_k(j\omega)| < 1$ for all k and ω. The intersection with the real axis becomes closer to $(-1, 0)$ for $k = 2$ as g_0 reaches 8. However, the intersection remains to the right of $(-1, 0)$ for any $g_0 > 0$ different from 8. The number of encirclements does not change regardless of the stability of the system because the Nyquist criterion is not used to count the number of RHP roots of the original polynomial. Instead, it is used to compare two polynomials with the same number of RHP roots.

CONCLUSIONS

This article provides an explanation and two short proofs of the Routh–Hurwitz criterion. The proofs were based on results presented in the literature after the original work of Routh. The author hopes that this tutorial presentation will be valuable in satisfying the curiosity of motivated students and their teachers, while providing interesting examples of application of root locus plots and the Nyquist criterion.

> **The key property implies that imaginary roots remain invariant at every step of the procedure.**

AUTHOR INFORMATION

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