ECE 486: Control Systems

Lecture 2B: Convolution Integral

Key Takeaways

For LTI systems, the response from zero initial conditions is given by an integral involving the input signal and the impulse response. This is known as the convolution integral.

The derivation relies on two properties of LTI systems: the principle of superposition and time-invariance.

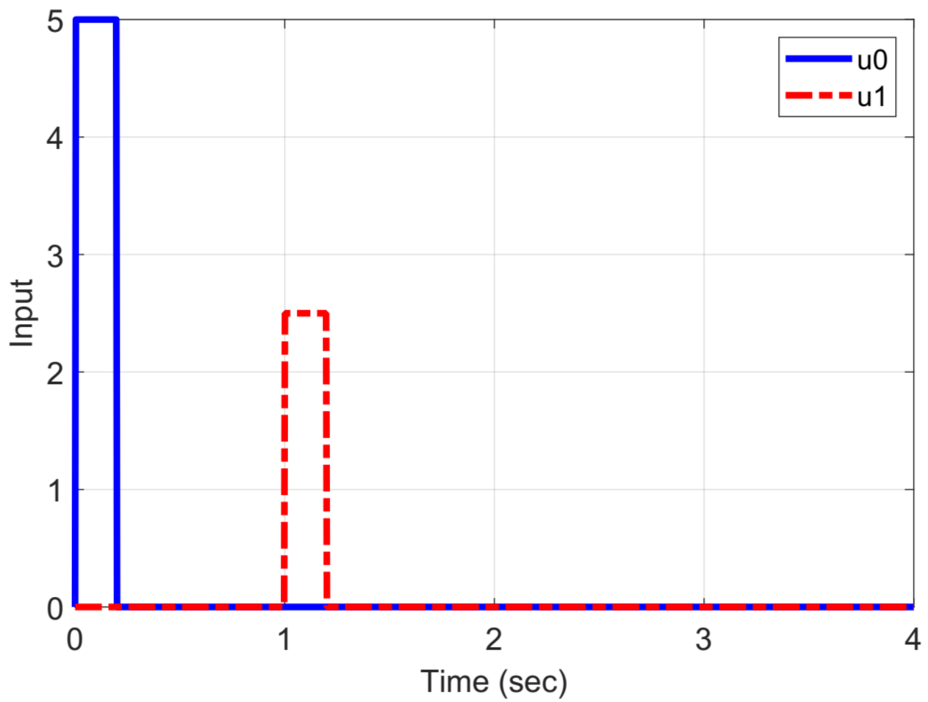
Superposition and Time-Invariance

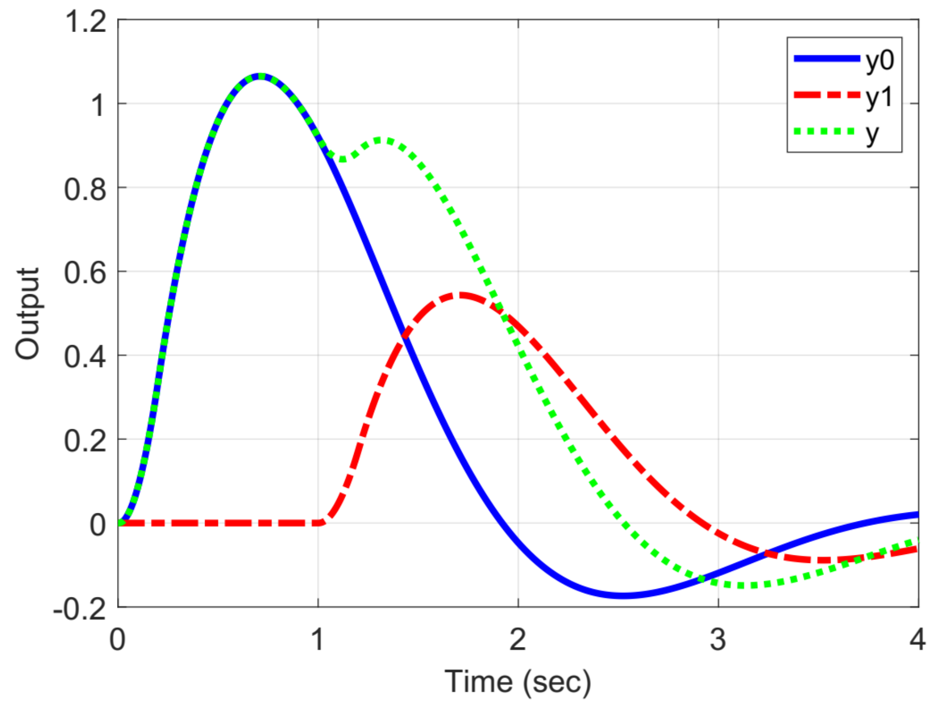
Consider the system $\ddot{y} + 2\dot{y} + 4y = 4u$.

Let y_0 and y_1 be the response due to u_0 and u_1 from zero ICs.

Note that u_1 is a scaled and shifted version u_0. Thus y_1 is a scaled and shifted version y_0

Also, the response due to u_0+u_1 is equal to y_0+y_1.

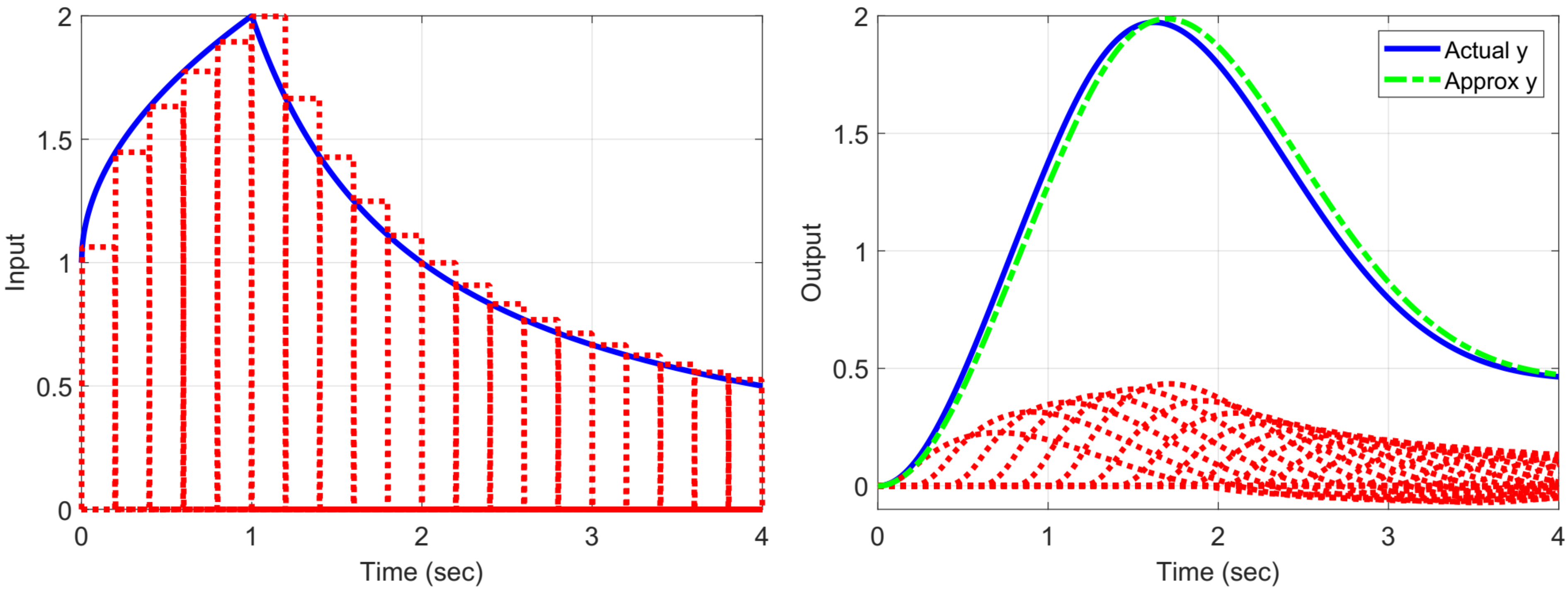




Pulse Response Approximation

Force the ODE with zero initial conditions with a short pulse input (width ε and height $1/\varepsilon$). Let $g_\varepsilon(t)$ denote the response.

We can approximate any input u(t) by a series of scaled and shifted pulses. Thus we can approximate the response due to u(t) from zero initial conditions by scaled/shifted versions of $g_\varepsilon(t)$.



Convolution Integral

As $\varepsilon \rightarrow 0$, the pulse converges to an idealized impulse $\delta(t)$ and the response $g_\varepsilon(t)$ converges to the impulse response $g(t)$.

The pulse response approximation converges to:

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau$$

This is called the **convolution integral** representation for the LTI system *G(s)*. This gives the output *y* when *G(s)* is forced by the input *u* from zero initial conditions.

The integrand only depends on the impulse response *g* and the input signal *u*. The formula also holds when integrating from 0 to infinity.