ECE 486: Control Systems

Lecture 2A: Ordinary Differential Equations (ODEs) & Linear Time-Invariant (LTI) Systems

Key Takeaways

The control design and analysis tools introduced in this course are primarily for systems modeled by linear ODEs with constant coefficients.

This lecture introduces:

- Nonlinear ODEs
- Linear ODEs with constant coefficients
- Principle of superposition
- Time invariance

Nonlinear ODEs

- Ordinary differential equations (ODEs) can be used to model the dynamics from an input *u* to an output *y*.
- An n^{th} order nonlinear ODE with initial conditions:

$$y^{[n]}(t) = f(y(t), \dot{y}(t), \ldots, y^{[n-1]}(t), u(t), \dot{u}(t), \ldots, u^{[m]}(t))$$

$$y(0) = y_0;\ \dot{y}(0) = \dot{y}_0;\ \ldots;\ y^{[n-1]}(0) = y_0^{[n-1]}$$

- Models can be constructed from physical laws or using experimental data
- We will not focus on constructing these models.
- Some examples are covered in Prof. Lessard’s notes.

Linear ODEs with Constant Coefficients

- An n^{th} order linear ODE with constant coefficients

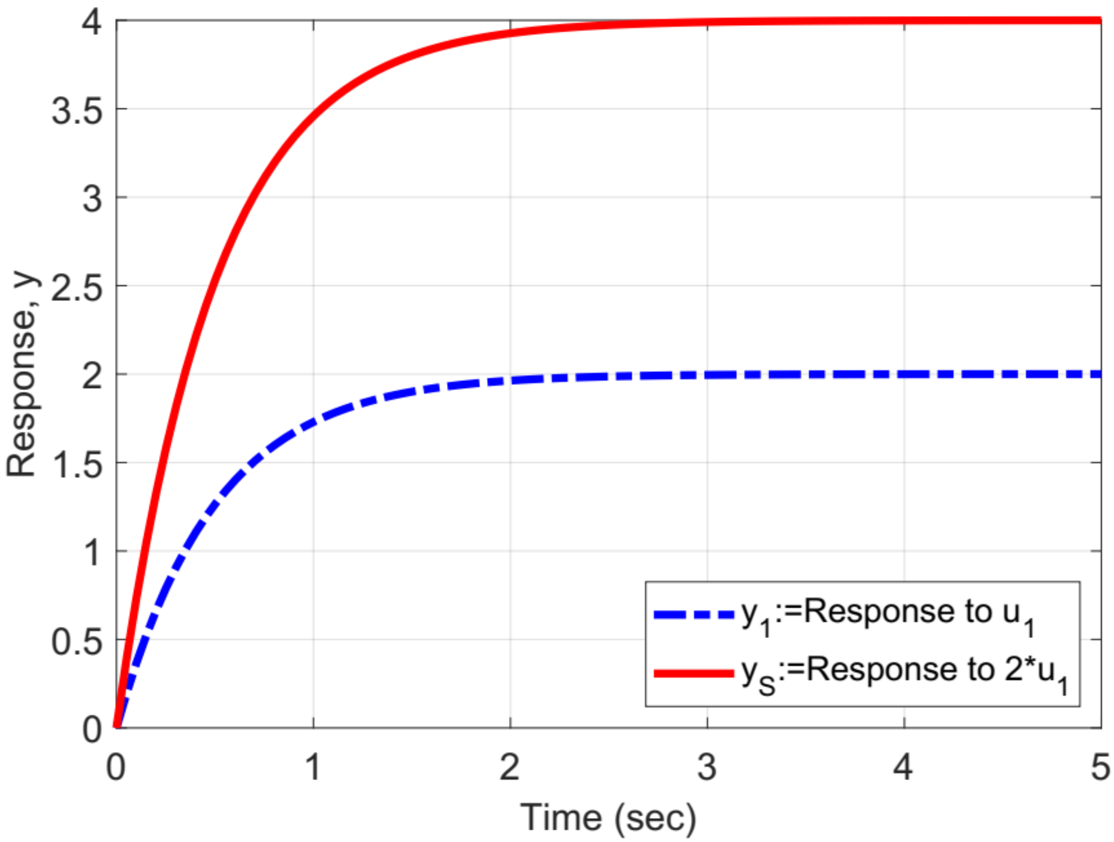
$$a_n y^{[n]}(t) + a_{n-1} y^{[n-1]}(t) + \cdots + a_1 \dot{y}(t) + a_0 y(t) = b_m u^{[m]}(t) + \cdots + b_1 \dot{u}(t) + b_0 u(t)$$

$$\text{ICs: } y(0) = y_0;\ \dot{y}(0) = \dot{y}_0;\ \ldots;\ y^{[n-1]}(0) = y_0^{[n-1]}$$

- Proper if $m \leq n$ and strictly proper if $m < n$
- Linear models often arise by approximating a nonlinear model. This step is called linearization and will be covered later in the course.
- Two key properties: Linearity & Time-Invariance

Principle of Superposition

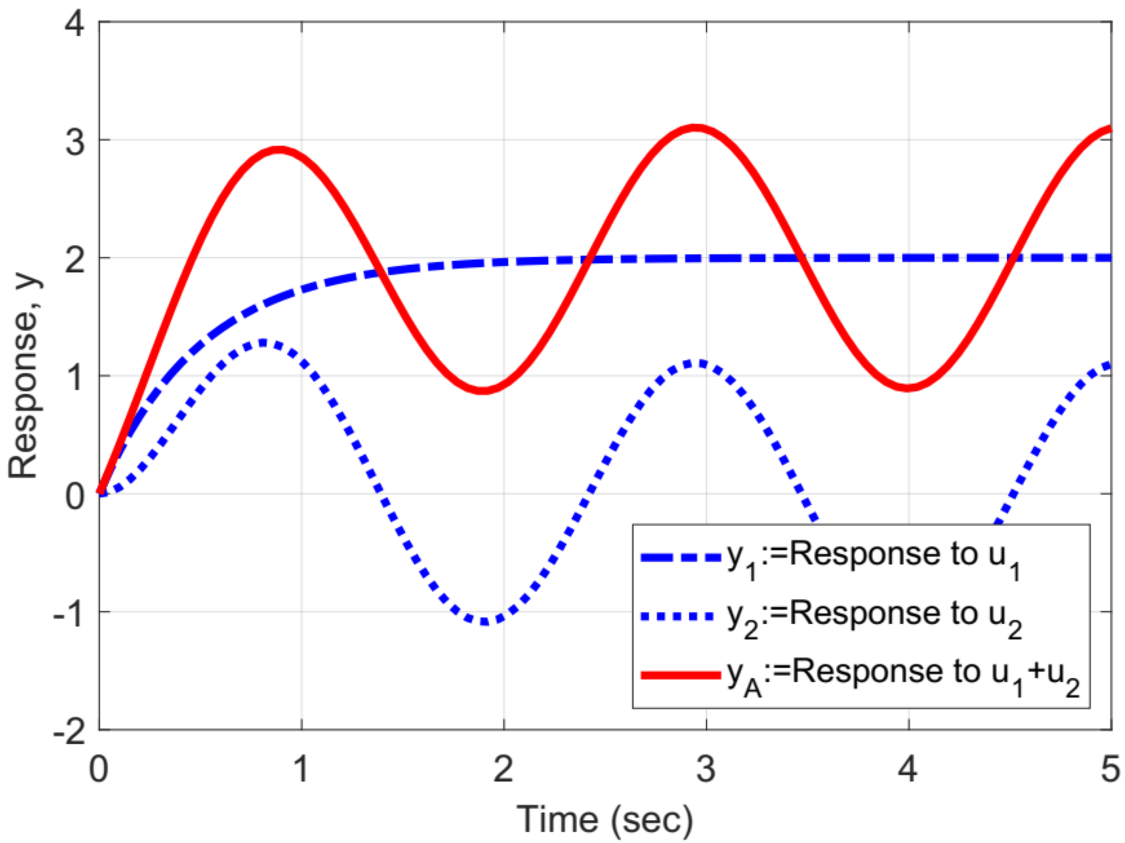
- Scaling: If the input u_1 generates output y_1 (with zero IC) then the input $c\,u_1$ generates $c\,y_1$ for any constant c.



$$\dot{y}(t) + 2y(t) = 4u(t)$$
with $y(0) = 0$
and $u_1(t) = 1$.

Principle of Superposition

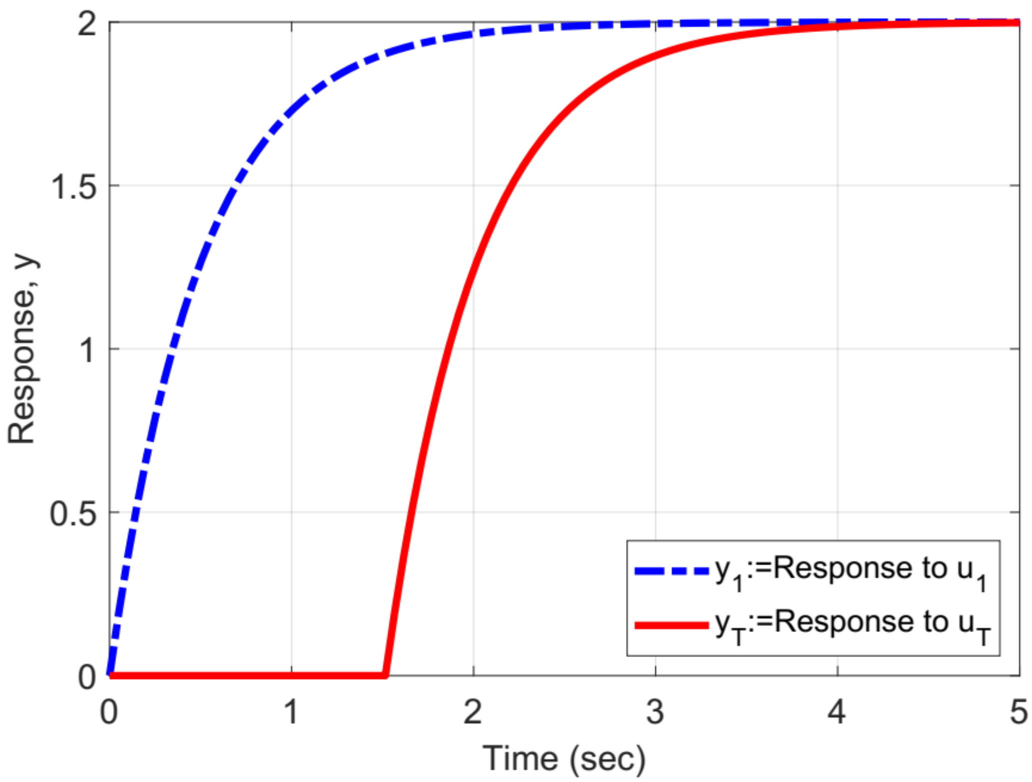
- Additivity: Suppose the inputs u_1 and u_2 generate outputs y_1 and y_2 (with zero IC). Then the input u_1+u_2 generates $y_1 + y_2$.



$\dot{y}(t) + 2y(t) = 4u(t)$ with $y(0) = 0$. Inputs are $u_1(t) = 1$ and $u_2(t) = \sin(3t)$.

Time-Invariance

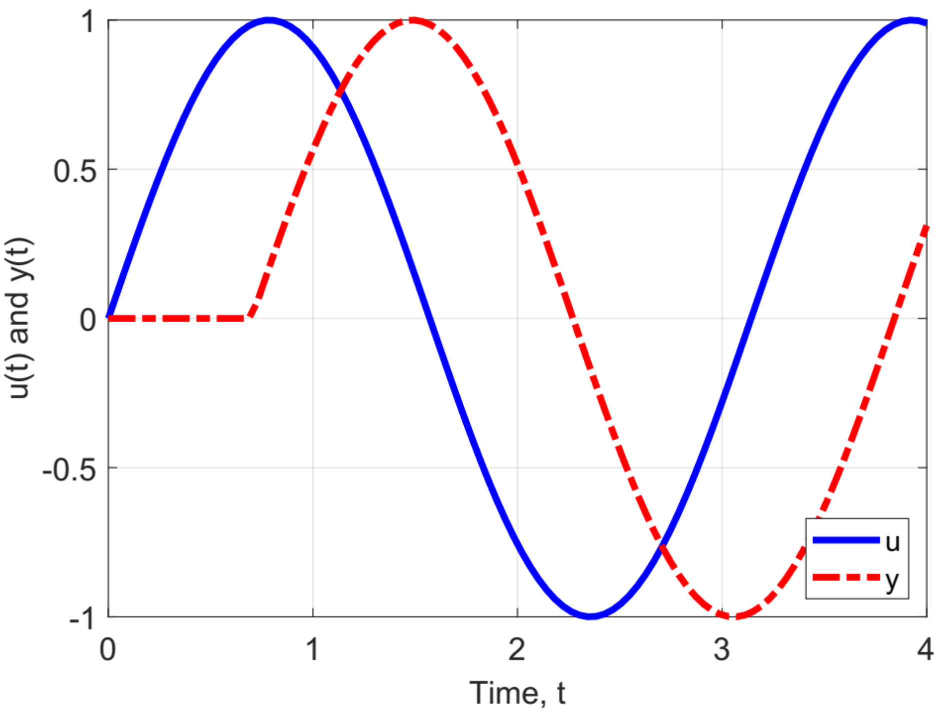
- Time-Invariance: Suppose the inputs u_1 generate the output y_1 (with zero IC). Shifting the input by T seconds will shift the response by T seconds.

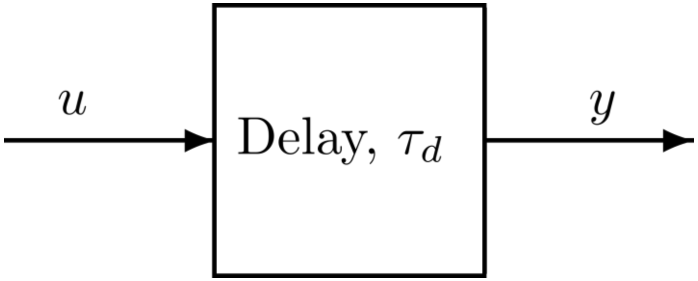


$\dot{y}(t) + 2y(t) = 4u(t)$
with $y(0) = 0$.
Inputs are $u_1(t) = 1$
and $u_T(t) := u_1(t - T)$.
Time shift is $T = 1.5\text{sec}$.

Time Delays

- Time-Delay: System that shifts (delays) the output relative to the input.
  - Delays satisfy superposition / time-invariance
  - Used to model implementation effects, e.g. computation time.





Delay: $\tau_d = 0.7\text{sec}$

Input: $u(t) = \sin(2t)$

Output:

$$y(t) = \begin{cases} 0 & 0 \le t < 0.7 \\ \sin(2(t-0.7)) & t \ge 0.7 \end{cases}$$