ECE486: Control Systems

- **Lecture 11A:** Introduction to Root Locus design method

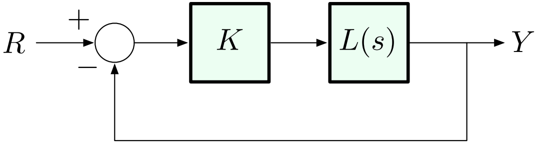
Goal: introduce the Root Locus method as a way of visualizing the locations of closed-loop poles of a given system as some parameter is varied.

Reading: FPE, Chapter 5

The Root Locus Design Method

(invented by Walter R. Evans in 1948)

Consider this unity feedback configuration:

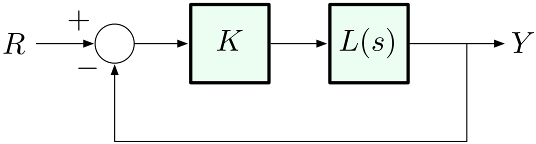


where

- K is a constant gain
- $L(s) = \dfrac{b(s)}{a(s)}$, where $a(s)$ and $b(s)$ are some polynomials

Problem: How to choose K to stabilize the closed-loop system?

The Root Locus Design Method



Closed-loop transfer function: $\dfrac{Y}{R} = \dfrac{KL(s)}{1+KL(s)}, \;\; L(s) = \dfrac{b(s)}{a(s)}$

Closed loop poles are solutions of:

$$1 + KL(s) = 0 \qquad \Leftrightarrow \qquad L(s) = -\frac{1}{K}$$

$$\Updownarrow$$

$$1 + \frac{Kb(s)}{a(s)} = 0$$

$$\Updownarrow$$

$$\underbrace{a(s) + Kb(s)}_{\substack{\text{characteristic}\\ \text{polynomial}}} = 0 \qquad \text{characteristic equation}$$

A Comment on Change of Notation

Note the change of notation:

$$\text{from } G(s) = \frac{q(s)}{p(s)} \quad \text{to } L(s) = \frac{b(s)}{a(s)}$$

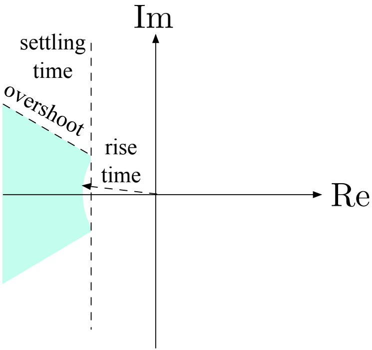
— the RL method is quite general, so $L(s)$ is not necessarily the *plant* transfer function, and K is not necessary *feedback gain* (could be *any parameter*).

E.g., $L(s)$ and K may be related to plant transfer function and feedback gain through some transformation.

As long as we can represent the poles of the closed-loop transfer function as roots of the equation $1 + KL(s) = 0$ for *some choice* of K and $L(s)$, we can apply the RL method.

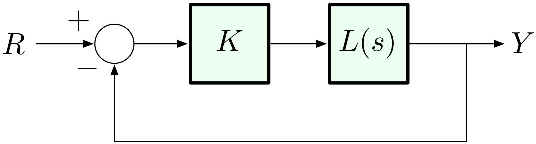
Towards Quantitative Characterization of Stability

Qualitative description of stability: Routh test gives us a range of K to guarantee stability.



For what values of K do we best satisfy given design specs?

Root Locus and Quantitative Stability



Closed-loop transfer function: $\frac{Y}{R} = \frac{KL(s)}{1+KL(s)}, \; L(s) = \frac{b(s)}{a(s)}$

For what values of K do we best satisfy given design specs?

Specs are encoded in pole locations, so:

The *root locus* for $1 + KL(s)$ is the set of all closed-loop poles, i.e., the roots of

$$1 + KL(s) = 0,$$

as K varies from 0 to ∞.

A Simple Example

$$L(s) = \frac{1}{s^2 + s} \qquad b(s) = 1,\ a(s) = s^2 + s$$

Characteristic equation: $\qquad a(s) + Kb(s) = 0$

$$s^2 + s + K = 0$$

Here, we can just use the quadratic formula:

$$s = -\frac{1 \pm \sqrt{1 - 4K}}{2} = -\frac{1}{2} \pm \frac{\sqrt{1 - 4K}}{2}$$

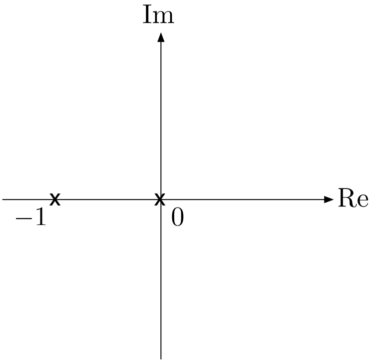
$$\text{Root locus} = \left\{ -\frac{1}{2} \pm \frac{\sqrt{1 - 4K}}{2} \ : 0 \le K < \infty \right\} \subset \mathbb{C}$$

Example, continued

$$\text{Root locus} = \left\{ -\frac{1}{2} \pm \frac{\sqrt{1-4K}}{2} \ : 0 \le K < \infty \right\} \subset \mathbb{C}$$

Let's plot it in the s-plane:

- start at $K = 0$ the roots are $-\frac{1}{2} \pm \frac{1}{2} \equiv -1, 0$
  note: these are poles of L (open-loop poles)

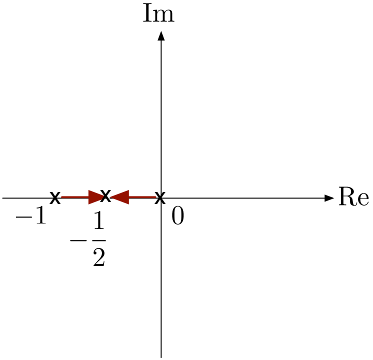


Example, continued

$$\text{Root locus: } \left\{ -\frac{1}{2} \pm \frac{\sqrt{1-4K}}{2} \ : 0 \le K < \infty \right\} \subset \mathbb{C}$$

- as K increases from 0, the poles start to move

$$\begin{aligned} 1 - 4K > 0 \qquad &\Longrightarrow \ 2 \text{ real roots} \\ K = 1/4 \qquad &\Longrightarrow 1 \text{ real root } s = -1/2 \end{aligned}$$

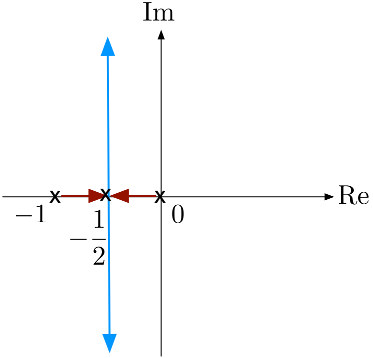


Example, continued

$$\text{Root locus: } \left\{ -\frac{1}{2} \pm \frac{\sqrt{1-4K}}{2} \ : 0 \le K < \infty \right\} \subset \mathbb{C}$$

- as K increases from 0, the poles start to move

$$K > 1/4 \qquad \Longrightarrow \text{ 2 complex roots with } \mathrm{Re}(s) = -1/2$$

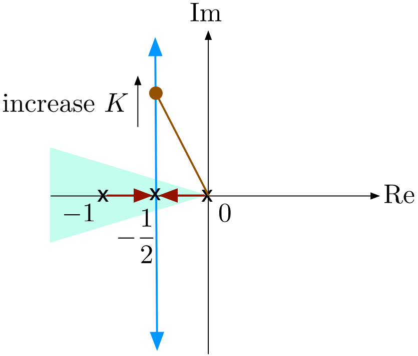


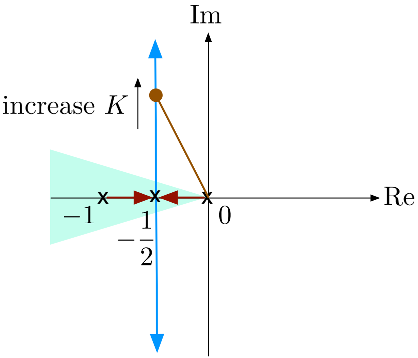
($s = -1/2$ is the *point of breakaway* from the real axis)

Example, continued

Compare this to admissible regions for given specs:

$t_s \approx \dfrac{3}{\sigma}$	want σ large, can only have $\sigma = \dfrac{1}{2}$ $(t_s = 6)$
$t_r \approx \dfrac{1.8}{\omega_n}$	want ω_n large $\Longrightarrow$ want K large
M_p	want to be inside the shaded region $\Longrightarrow$ want K small





Thus, the root locus helps us *visualize the trade-off* between all the specs in terms of K.

However, for order > 2, there will generally be no direct formula for the closed-loop poles as a function of K.

Our goal: develop simple rules for (approximately) sketching the root locus in the general case.