Plan of the Lecture

► Review: observability; Luenberger observer and state estimation error.

► Today’s topic: joint observer and controller design: dynamic output feedback.

**Goal:** learn how to design an observer and a controller to achieve accurate closed-loop pole placement.

**Reading:** FPE, Chapter 7
Is Full State Feedback Always Available?

In a typical system, measurements are provided by sensors:

\[ \frac{d}{dt} x = A x + B u \]
\[ y = x \]

Full state feedback \( u = -Kx \) is **not implementable**!!

In that case, an observer is used to estimate the state \( x \):

\[ y = C \hat{x} \]

\( \hat{x} \) is an estimate of \( x \).

If \( C(A, B) \) is full rank, then we can do ordinary pole placement with full feedback.
State Estimation Using an Observer

If the system is observable, the state estimate \( \hat{x} \) is asymptotically accurate:

\[
\|\hat{x}(t) - x(t)\| = \sqrt{\sum_{i=1}^{n} (\hat{x}_i(t) - x_i(t))^2} \xrightarrow{t \to \infty} 0
\]

If we are successful, then we can try estimated state feedback:

\[
u = -K\hat{x}
\]

Can we close the loop using \( \hat{x} \) instead of \( x \)?
Observability

Consider a single-output system \((y \in \mathbb{R})\):

\[ \dot{x} = Ax + Bu, \quad y = Cx \quad x \in \mathbb{R}^n \]

The Observability Matrix is defined as

\[ \mathcal{O}(A, C) = 
\begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{n-1}
\end{bmatrix} \]

We say that the above system is observable if its observability matrix \(\mathcal{O}(A, C)\) is invertible.

(This definition is only true for the single-output case; the multiple-output case involves the rank of \(\mathcal{O}(A, C)\).)
Observer Canonical Form

A single-output state-space model

\[ \dot{x} = Ax + Bu, \quad y = Cx \]

is said to be in Observer Canonical Form (OCF) if the matrices

\[ A, C \]

are of the form

\[
A = \begin{pmatrix}
0 & 0 & \ldots & 0 & 0 & * \\
1 & 0 & \ldots & 0 & 0 & * \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 & * \\
0 & 0 & \ldots & 0 & 1 & * \\
\end{pmatrix}, \quad C = (0 \ 0 \ \ldots \ 0 \ 1)
\]

Fact: A system in OCF is always observable!!

(The proof of this for \( n > 2 \) uses the Jordan canonical form, we will not worry about this.)
The Luenberger Observer

System: \[
\dot{x} = Ax + Bu \\
y = Cx
\]

Observer: \[
\dot{\hat{x}} = (A - LC)\hat{x} + Ly + Bu
\]

What happens to state estimation error \( e = x - \hat{x} \) as \( t \to \infty \)?

\[
\dot{e} = (A - LC)e
\]

Does \( e(t) \) converge to zero in some sense?
The Luenberger Observer

System:
\[ \dot{x} = Ax \]
\[ y = Cx \]

Observer:
\[ \dot{\hat{x}} = (A - LC)\hat{x} + Ly \]

Error:
\[ \dot{e} = (A - LC)e \]

Recall our assumption that \( A - LC \) is Hurwitz (all eigenvalues are in LHP). This implies that

\[ \|x(t) - \hat{x}(t)\|^2 = \|e(t)\|^2 = \sum_{i=1}^{n} |e_i(t)|^2 \xrightarrow{t \to \infty} 0 \]

at an exponential rate, determined by the eigenvalues of \( A - LC \).

For fast convergence, want eigenvalues of \( A - LC \) far into LHP!!
Observability and Estimation Error

Fact: If the system
\[ \dot{x} = Ax, \quad y = Cx \]
is observable, then we can arbitrarily assign eigenvalues of \( A - LC \) by a suitable choice of the output injection matrix \( L \).

This is similar to the fact that controllability implies arbitrary closed-loop pole placement by state feedback.

In fact, these two facts are closely related because CCF is dual to OCF.
Combining Full-State Feedback with an Observer

- So far, we have focused on autonomous systems \((u = 0)\).
- What about nonzero inputs?

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}
\]

— assume \((A, B)\) is controllable and \((A, C)\) is observable.

- Today, we will learn how to use an observer together with estimated state feedback to (approximately) place closed-loop poles.

\[u = -K\hat{x}\]
Combining Full-State Feedback with an Observer

Consider

\[ \dot{x} = Ax + Bu \]

\[ y = Cx \]

where \((A, B)\) is controllable and \((A, C)\) is observable.

We know how to find \(K\), such that \(A - BK\) has desired eigenvalues (controller poles).

Since we do not have access to \(x\), we must design an observer. But this time, we need a slight modification because of the \(Bu\) term.
Observer in the Presence of Control Input

- Let’s see what goes wrong when we use the old approach:

\[
\dot{\hat{x}} = (A - LC)\hat{x} + Ly
\]

- For the estimation error \( e = x - \hat{x} \), we have

\[
\dot{e} = \dot{x} - \dot{\hat{x}}
= Ax + Bu - [(A - LC)\hat{x} + LCx]
= (A - LC)e + Bu
\]

- \textbf{Idea:} since \( u \) is a signal we can access, let’s use it as an input to the observer to cancel the \( Bu \) term from \( \dot{x} \).

- \textbf{Modified observer:}

\[
\begin{align*}
\dot{\hat{x}} &= (A - LC)\hat{x} + Ly + Bu \\
\dot{e} &= \dot{x} - \dot{\hat{x}} \\
&= Ax + Bu - [(A - LC)\hat{x} + LCx + Bu] \\
&= (A - LC)e
\end{align*}
\]

regardless of \( u \)
Observer and Controller

System: \[ \dot{x} = Ax + Bu \]
\[ y = Cx \]

Observer: \[ \hat{x} = (A - LC)\hat{x} + Ly + Bu \]

Error: \[ \dot{e} = (A - LC)e \]

- By observability, we can arbitrarily assign \( \text{eig}(A - LC) \); these should be farther into LHP than desired controller poles.

Controller: \[ u = -K\hat{x} \quad \text{(estimated state feedback)} \]

- By controllability, we can arbitrarily assign \( \text{eig}(A - BK) \).
Observer and Controller

System:
\[ \dot{x} = Ax + Bu \]
\[ y = Cx \]

Observer:
\[ \dot{\hat{x}} = (A - LC)\hat{x} + Ly + Bu \]

Controller:
\[ u = -K\hat{x} \]

The overall observer-controller system is:
\[ \dot{\hat{x}} = (A - LC)\hat{x} + Ly + B(-K\hat{x}) = u \]
\[ u = -(A - LC - BK)\hat{x} + Ly \]

This is a dynamical system with input \( y \) and output \( u \).
Dynamic Output Feedback

\[
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx \\
\dot{\hat{x}} &= (A - LC - BK)\hat{x} + Ly \\
u &= -K\hat{x}
\end{aligned}
\]
Dynamic Output Feedback

\[
\dot{\hat{x}} = (A - LC - BK)\hat{x} + Ly, \quad u = -K\hat{x}
\]

Controller transfer function (from \(y\) to \(u\)):

\[
s\hat{X} = (A - LC - BK)\hat{X} + LY, \quad U = -K\hat{X}
\]

\[
U = -K(I_s - A + LC + BK)^{-1}LY = D(s)
\]
Dynamic Output Feedback: Does It Work?

Summarizing:

- When $y = x$, full state feedback $u = -Kx$ achieves desired pole placement.
- How do we know that $u = -K\hat{x}$ achieves similar objectives?

Here is our overall closed-loop system:

\[
\begin{align*}
\dot{x} &= Ax - BK\hat{x} \\
\dot{\hat{x}} &= (A - LC - BK)\hat{x} + LCx
\end{align*}
\]

We can write it in block matrix form:

\[
\begin{pmatrix}
\dot{x} \\
\dot{\hat{x}}
\end{pmatrix} =
\begin{pmatrix}
A & -BK \\
LC & A - LC - BK
\end{pmatrix}
\begin{pmatrix}
x \\
\hat{x}
\end{pmatrix}
\]

How do we relate this to “nominal” behavior, $A - BK$?
Dynamic Output Feedback

\[
\begin{pmatrix}
\dot{x} \\
\dot{\hat{x}}
\end{pmatrix} = 
\begin{pmatrix}
A & -BK \\
LC & A - LC - BK
\end{pmatrix} 
\begin{pmatrix}
x \\
\hat{x}
\end{pmatrix}
\]

Let us transform to new coordinates:

\[
\begin{pmatrix}
x \\
\hat{x}
\end{pmatrix} \mapsto \begin{pmatrix}
x \\
e
\end{pmatrix} = 
\begin{pmatrix}
x \\
x - \hat{x}
\end{pmatrix} = 
\begin{pmatrix}
I & 0 \\
I & -I
\end{pmatrix} 
\begin{pmatrix}
x \\
\hat{x}
\end{pmatrix}
\]

Two key observations:

- \( T \) is invertible, so the new representation is equivalent to the old one
- in the new coordinates, we have

\[
\dot{x} = Ax - BK\hat{x} + BK(x - \hat{x}) = (A - BK)x + BK(x - \hat{x}) = (A - BK)x + BKe
\]

\[
\dot{e} = (A - LC)e
\]
The Main Result: Separation Principle

So now we can write

\[
\begin{bmatrix}
\dot{x} \\
\dot{e}
\end{bmatrix} = \begin{bmatrix}
A - BK & BK \\
0 & A - LC
\end{bmatrix}
\begin{bmatrix}
x \\
e
\end{bmatrix}
\]

upper triangular matrix

The closed-loop characteristic polynomial is

\[
\text{det}\left( I_s - A + BK \begin{bmatrix} -BK \\ 0 \end{bmatrix} I_s - A + LC \right) = \text{det} (I_s - A + BK) \cdot \text{det} (I_s - A + LC)
\]

Separation principle. The closed-loop eigenvalues are:

\[
\{ \text{controller poles (roots of } \text{det}(I_s - A + BK)) \} \cup \{ \text{observer poles (roots of } \text{det}(I_s - A + LC)) \}
\]

— this holds only for linear systems!!
Separation Principle

Separation principle. The closed-loop eigenvalues are:

\[
\{\text{controller poles (roots of } \det(I s - A + BK))\}\}
\cup \{\text{observer poles (roots of } \det(I s - A + LC))\}\}

— this holds only for linear systems!!

Moral of the story:

- If we choose observer poles to be several times faster than the controller poles (e.g., 2–5 times), then the controller poles will be dominant.
- Dynamic output feedback gives essentially the same performance as (nonimplementable) full-state feedback — provided observer poles are far enough into LHP.
- Remember: the system must be controllable and observable!!