Plan of the Lecture

- **Review:** Nyquist stability criterion
- **Today’s topic:** Nyquist stability criterion (more examples); phase and gain margins from Nyquist plots.

**Goal:** explore more examples of the Nyquist criterion in action.

**Reading:** FPE, Chapter 6
Review: Nyquist Plot

Consider an arbitrary transfer function $H$.

Nyquist plot: $\text{Im } H(j\omega)$ vs. $\text{Re } H(j\omega)$ as $\omega$ varies from $-\infty$ to $\infty$.
Review: Nyquist Stability Criterion

Goal: count the number of RHP poles (if any) of the closed-loop transfer function

\[
\frac{KG(s)}{1 + KG(s)} \quad \text{RHP \ pole}\]

based on frequency-domain characteristics of the plant transfer function \(G(s)\)
Nyquist Theorem (1928) Assume that $G(s)$ has no poles on the imaginary axis*, and that its Nyquist plot does not pass through the point $-1/K$. Then

$$N = Z - P$$

$$\#(\text{Circum} \ of \ -1/K \ \text{by Nyquist plot of } G(s))$$

$$= \#(\text{RHP closed-loop poles}) - \#(\text{RHP open-loop poles})$$

* Easy to fix: draw an infinitesimally small circular path that goes around the pole and stays in RHP
The Nyquist Stability Criterion

Nyquist Stability Criterion. Under the assumptions of the Nyquist theorem, the closed-loop system (at a given gain $K$) is stable if and only if the Nyquist plot of $G(s)$ encircles the point $-1/K$ $P$ times counterclockwise, where $P$ is the number of unstable (RHP) open-loop poles of $G(s)$. 
Applying the Nyquist Criterion

Workflow:

Bode $M$ and $\phi$-plots $\rightarrow$ Nyquist plot

Advantages of Nyquist over Routh–Hurwitz

- can work directly with experimental frequency response data (e.g., if we have the Bode plot based on measurements, but do not know the transfer function)
- less computational, more geometric (came 55 years after Routh)
Example 1 (From Last Lecture)

\[ G(s) = \frac{1}{(s + 1)(s + 2)} \quad \text{(no open-loop RHP poles)} \]

Characteristic equation:

\[ (s + 1)(s + 2) + K = 0 \iff s^2 + 3s + K + 2 = 0 \]

From Routh, we already know that the closed-loop system is stable for \( K > -2 \).

We will now reproduce this answer using the Nyquist criterion.

\[ N = 2 - P \]

\[ \text{CL poles} \quad \text{OL poles} \]

\[ \text{N cannot be encircled by \( N \) poles.} \]
Example 1

\[ G(s) = \frac{1}{(s + 1)(s + 2)} \]  
(no open-loop RHP poles)

Strategy:

- Start with the Bode plot of \( G \)
- Use the Bode plot to graph \( \text{Im } G(j\omega) \) vs. \( \text{Re } G(j\omega) \) for \( 0 \leq \omega < \infty \)
- This gives only a portion of the entire Nyquist plot

\[ (\text{Re } G(j\omega), \text{Im } G(j\omega)) , \quad -\infty < \omega < \infty \]

- Symmetry:

\[ G(-j\omega) = \overline{G(j\omega)} \]

— Nyquist plots are always symmetric w.r.t. the real axis!!
Example 1

\[ G(s) = \frac{1}{(s + 1)(s + 2)} \]

(no open-loop RHP poles)

Bode plot:

Nyquist plot:
Example 1: Applying the Nyquist Criterion

\[ G(s) = \frac{1}{(s + 1)(s + 2)} \]  
(no open-loop RHP poles)

Nyquist plot:

\[ \#(\bigcirc \text{ of } -1/K) = \#(\text{RHP CL poles}) - \#(\text{RHP OL poles}) = 0 \]

\[ \implies K \in \mathbb{R} \text{ is stabilizing if and only if} \]

\[ \#(\bigcirc \text{ of } -1/K) = 0 \]

- If \( K > 0 \), \( \#(\bigcirc \text{ of } -1/K) = 0 \)
- If \( 0 < -1/K < 1/2 \), \( \#(\bigcirc \text{ of } -1/K) > 0 \implies \) closed-loop stable for \( K > -2 \)
Example 2

\( G(s) = \frac{1}{(s-1)(s^2 + 2s + 3)} = \frac{1}{s^3 + s^2 + s - 3} \)

\( \#(\text{RHP open-loop poles}) = 1 \) at \( s = 1 \)

Routh: the characteristic polynomial is

\[ s^3 + s^2 + s + K - 3 \]

— 3rd degree

— stable if and only if \( K - 3 > 0 \) and \( 1 > K - 3 \).

Stability range: \( 3 < K < 4 \)

Let’s see how to spot this using the Nyquist criterion ...
Example 2

\[ G(s) = \frac{1}{(s - 1)(s^2 + 2s + 3)} \]

(1 open-loop RHP pole)

Bode plot:

Nyquist plot:

\[ G(j\omega) = \frac{1}{(j\omega - 1) \left(-\omega^2 + 2\omega + 3\right)} \]

- for \( \omega = 0 \) \( M = 1/3, \phi = -180^\circ \)
- for \( \omega = 1 \) \( M = 1/4, \phi = -180^\circ \)
- for \( \omega \to \infty \) \( M \to 0, \phi \to -270^\circ \)
Example 2: Applying the Nyquist Criterion

\[ G(s) = \frac{1}{(s - 1)(s^2 + 2s + 3)} \]

(1 open-loop RHP pole)

Nyquist plot:

\[ \#(\succ of -1/K) = \#(\text{CL poles}) - \#(\text{OL poles}) = 1 \]

\( K \in \mathbb{R} \) is stabilizing if and only if

\[ \#(\succ of -1/K) = -1 \]

Which points \(-1/K\) are encircled once \(\succ\) by this Nyquist plot?

only \(-1/3 < -1/K < -1/4\) \(\implies 3 < K < 4\)
Example 2: Nyquist Criterion and Phase Margin

Closed-loop stability range for $G(s) = \frac{1}{(s - 1)(s^2 + 2s + 3)}$ is $3 < K < 4$ (using either Routh or Nyquist).

We can interpret this in terms of phase margin:

So, in this case, stability $\iff$ $\text{PM} > 0$ (typical case).
Example 3

\[ G(s) = \frac{s - 1}{(s + 2)(s^2 - s + 1)} = \frac{s - 1}{s^3 + s^2 - s + 2} \]

Non-MP zero at \( s = -1 \), 2 unstable OC poles.

Open-loop poles:

\[ s = -2 \]
\[ s^2 - s + 1 = 0 \]
\[ \left( s - \frac{1}{2} \right)^2 + \frac{3}{4} = 0 \]
\[ s = \frac{1}{2} \pm j \frac{\sqrt{3}}{2} \]

\( \therefore \) 2 RHP poles
Example 3

\[ G(s) = \frac{s - 1}{(s + 2)(s^2 - s + 1)} = \frac{s - 1}{s^3 + s^2 - s + 2} \]

Routh:

char. poly. \[ s^3 + s^2 - s + 2 + K(s - 1) \]

\[ s^2 + s^2 + (K - 1)s + 2 - K \] (3rd-order)

— stable if and only if

\[ K - 1 > 0 \quad \Rightarrow \quad K > 1 \]
\[ 2 - K > 0 \quad \Rightarrow \quad K < 2 \]
\[ K - 1 > 2 - K \]

— stability range is \( \frac{3}{2} < K < 2 \)
Example 3

\[ G(s) = \frac{s - 1}{(s + 2)(s^2 - s + 1)} \]  

(2 open-loop RHP poles)

Bode plot (tricky, RHP poles/zeros)

\[ \phi = 180^\circ \text{ when:} \]

- \( \omega = 0 \) and \( \omega \to 0 \)
- \( \omega = 1/\sqrt{2} \):

\[
\left. \frac{j\omega - 1}{(j\omega - 1)((j\omega)^2 - j\omega + 1)} \right|_{\omega=1/\sqrt{2}} = \frac{j}{\sqrt{2}} - 1 \\
= \frac{j}{\sqrt{2}} - 1 \\
= \frac{j}{\sqrt{2}} - 1 \\
= \frac{j}{\sqrt{2}} - 1 \\
= -\frac{3}{2} \left( \frac{j}{\sqrt{2}} - 1 \right) = -\frac{3}{2}
\]

(need to guess this, e.g., by mouseclicking in Matlab)
Example 3

\[ G(s) = \frac{s - 1}{s^3 + s^2 - s + 2} \]

(2 open-loop RHP poles)

Bode plot:

Nyquist plot:

\[
\begin{align*}
\omega = 0 & \quad M = 1/2, \phi = 180^\circ \\
\omega = 1/\sqrt{2} & \quad M = 2/3, \phi = 180^\circ \\
\omega \to \infty & \quad M \to 0, \phi \to 180^\circ
\end{align*}
\]
Example 3: Applying the Nyquist Criterion

\[ G(s) = \frac{s - 1}{s^3 + s^2 - s + 2} \]

(2 open-loop RHP poles)

\[ \#(\bigcirc \text{ of } -1/K) = -2 \]

\[ K \in \mathbb{R} \text{ is stabilizing if and only if} \]

\[ \#(\bigcirc \text{ of } -1/K) = -2 \]

Which points \(-1/K\) are encircled twice \(\bigcirc\) by this Nyquist plot?

\[ \#(\bigcirc \text{ of } -1/K) = \#(\text{RHP CL poles}) - \#(\text{RHP OL poles}) = 2 \]

only \(-2/3 < -1/K < -1/2\)

\[ \implies \frac{3}{2} < K < 2 \]
Example 2: Nyquist Criterion and Phase Margin

CL stability range for $G(s) = \frac{s - 1}{s^3 + s^2 - s + 2} : K \in (3/2, 2)$

We can interpret this in terms of phase margin:

![Graph showing phase margin and Nyquist plot]

So, in this case, stability $\iff$ PM $< 0$ (atypical case; Nyquist criterion is the only way to resolve this ambiguity of Bode plots).
Stability Margins

How do we determine stability margins (GM & PM) from the Nyquist plot?

GM & PM are defined relative to a given \( K \), so consider Nyquist plot of \( KG(s) \) (we only draw the \( \omega > 0 \) portion):

How do we spot GM & PM?

- **GM** = \( 1/M_{180^\circ} \)
  - if we divide \( K \) by \( M_{180^\circ} \), then the Nyquist plot will pass through \((-1, 0)\), giving \( M = 1, \phi = 180^\circ \)

- **PM** = \( \phi \)
  - the phase difference from \( 180^\circ \) when \( M = 1 \)