Plan of the Lecture

▶ Review: control design using frequency response: PI/lead

▶ Today’s topic: control design using frequency response: PD/lag, PID/lead+lag

Goal: understand the effect of various types of controllers (PD/lead, PI/lag) on the closed-loop performance by reading the open-loop Bode plot; develop frequency-response techniques for shaping transient and steady-state response using dynamic compensation

Reading: FPE, Chapter 6
Assuming that $G(s)$ is *minimum-phase* (i.e., has no RHP zeros), we derived the following for the Bode plot of $KG(s)$:

<table>
<thead>
<tr>
<th></th>
<th>low freq.</th>
<th>real zero/pole</th>
<th>complex zero/pole</th>
</tr>
</thead>
<tbody>
<tr>
<td>mag. slope</td>
<td>$n$</td>
<td>up/down by 1</td>
<td>up/down by 2</td>
</tr>
<tr>
<td>phase</td>
<td>$n \times 90^\circ$</td>
<td>up/down by $90^\circ$</td>
<td>up/down by $180^\circ$</td>
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</table>

We can state this succinctly as follows:

**Gain-Phase Relationship.** Far enough from break-points,

$$\text{Phase} \approx \text{Magnitude Slope} \times 90^\circ$$
Bode’s Gain-Phase Relationship

Gain-Phase Relationship. Far enough from break-points,

\[ \text{Phase} \approx \text{Magnitude Slope} \times 90^\circ \]

This suggests the following rule of thumb:

\[ M = 1 \text{ want slope } = \frac{1}{0.01} \approx 100 \]

\[ \omega_c \text{ i.f. } M(\omega_c) = 1 \]

- \( M \) has slope \(-2\) at \( \omega_c \)
  \[ \Rightarrow \phi(\omega_c) = -180^\circ \]
  \[ \Rightarrow \text{bad (no PM)} \]

- \( M \) has slope \(-1\) at \( \omega_c \)
  \[ \Rightarrow \phi(\omega_c) = -90^\circ \]
  \[ \Rightarrow \text{good (PM = 90}^\circ\)]

— this is an important design guideline!!

(Similar considerations apply when \( M \)-plot has positive slope – depends on the t.f.)
Bode’s Gain-Phase Relationship suggests that we can shape the time response of the *closed-loop* system by choosing $K$ (or, more generally, a dynamic controller $KD(s)$) to tune the Phase Margin.

In particular, from the quantitative Gain-Phase Relationship,

$$\text{Magnitude slope}(\omega_c) = -1 \quad \implies \quad \text{Phase}(\omega_c) \approx -90^\circ$$

—which gives us PM of $90^\circ$ and consequently *good damping*. 
Lead Controller Design Using Frequency Response

General Procedure

1. Choose $K$ to get desired bandwidth spec w/o lead
2. Choose lead zero and pole to get desired PM
   - in general, we should first check PM with the $K$ from 1, w/o lead, to see how much more PM we need
3. Check design and iterate until specs are met.

This is an intuitive procedure, but it’s not very precise, requires trial & error.
Lag Compensation: Bode Plot

\[ D(s) = \frac{s + z}{s + \frac{z}{p}} = \frac{z}{p} \frac{s}{s + \frac{1}{p}} + 1, \quad z \gg p \]

so \( M \to 1 \) at high frequencies

\[ \frac{j\omega + z}{j\omega + p} \xrightarrow{\omega \to \infty} 1 \]

- subtracts phase, hence the term “phase lag”
Lag Compensation: Bode Plot

In general, s.s. tracking error is better as $M(\omega)$ is higher. A lag controller helps this. Odd $z/p$ helps M(\omega)!

\[ \frac{j\omega + z}{j\omega + p} \xrightarrow{\omega \to 0} \frac{z}{p} \]

\[ sR(s) \]

steady-state tracking error:

\[ e(\infty) = \left| \frac{sR(s)}{1 + D(s)G(s)} \right|_{s=0} \]

large $z/p \implies$ better s.s. tracking

- lag decreases $\omega_c \implies$ slows down time response (to compensate, adjust $K$ or add lead)

- **caution:** lead increases PM, but adding lag can undo this

- to mitigate this, choose both $z$ and $p$ very small, while maintaining desired ratio $z/p$
Example of lag design.

\[ G(s) = \frac{1}{(s + 0.2)(s + 0.5)} \]

Bode form

\[ \frac{10}{(\frac{s}{0.2} + 1)(\frac{s}{0.5} + 1)} \]

Objectives:

- \( \text{PM} \geq 60^\circ \)
- \( e(\infty) \leq 10\% \) for constant reference (closed-loop tracking error)

Strategy:

- we will use lag

\[ KD(s) = K \frac{s + z}{s + p}, \quad z \gg p \]

- \( z \) and \( p \) will be chosen to get good tracking
- PM will be shaped by choosing \( K \)
- this is different from what we did for lead (used \( p \) and \( z \) to shape PM, then chose \( K \) to get desired bandwidth spec)
Step 1: Choose $K$ to Shape PM

Check Bode plot of $G(s)$ to see how much PM it already has:

- from Matlab, $\omega_c \approx 1$
- PM $\approx 40^\circ$
- we want PM $= 60^\circ$

$\phi = -120^\circ$ at $\omega \approx 0.573$

$M = 2.16$

A conservative choice (to allow some slack) is $K = 1/2.16 = 0.4$, gives $\omega_c \approx 0.52$, PM $\approx 65^\circ$
Step 2: Choose $z$ & $p$ to Shape Tracking Error

So far: $KG(s) = \frac{0.4 \cdot 10}{(\frac{s}{0.2} + 1)(\frac{s}{0.5} + 1)}$

$$e(\infty) = \left. \frac{1}{1 + KG(s)} \right|_{s=0} = \frac{1}{1 + 4} = \frac{1}{5} = 20\% \text{ (too high)}$$

To have $e(\infty) \leq 10\%$, need $KD(0)G(0) \geq 9$:

$$e(\infty) = \frac{1}{1 + KD(0)G(0)} \leq \frac{1}{1 + 9} = 10\%.$$

So, we need

\[ D(0) = \left. \frac{s + z}{s + p} \right|_{s=0} = \frac{z}{p} \geq \frac{9}{4} = 2.25 \quad \text{— say, } z/p = 2.5 \]

Not to distort PM and $\omega_c$, let’s pick $z$ and $p$ an order of magnitude smaller than $\omega_c \approx 0.5$: $z = 0.05$, $p = 0.02$
Overall Design

- Set $k$ to have desired $w_c$ & $P_M$
- Set $3/5K$ at $e_{[0]} = 3\psi$

\[ G(s) = \frac{10}{\left(\frac{s}{0.2} + 1\right)\left(\frac{s}{0.5} + 1\right)} \]

Controller:

\[ KD(s) = 0.4 \frac{s + 0.05}{s + 0.02} \]

— the design still needs a bit of refinement ...
Lead & Lag Compensation

Let’s combine the advantages of PD/lead and PI/lag.

Back to our example: \( G(s) = \frac{10}{\left( \frac{s}{0.2} + 1 \right) \left( \frac{s}{0.5} + 1 \right)} \)

from Matlab, \( \omega_c \approx 1 \)

PM \( \approx 40^\circ \)

New objectives:

- \( \omega_{BW} \geq 2 \)
- PM \( \geq 60^\circ \)
- \( e(\infty) \leq 1\% \) for const. ref.
Lead & Lag Compensation

What we got before, with lag only:

- Improved PM by adjusting $K$ to decrease $\omega_c$.
- This gave $\omega_c \approx 0.5$, whereas now we want a larger $\omega_c$
  (recall: $\omega_{BW} \in [\omega_c, 2\omega_c]$, so $\omega_c = 0.5$ is too small)

So: we need to reshape the phase curve using lead.
Step 1. Choose $K$ to get $\omega_c \approx 2$ (before lead)

Using Matlab, can check:

at $\omega = 2$, $M \approx 0.24$ (with $K = 1$)

— need $K = \frac{1}{0.24} \approx 4.1667$

— choose $K = 4$

(gives $\omega_c$ slightly $< 2$, but still ok).
Lead & Lag Compensation

Step 2. Decide how much phase lead is needed, and choose $z_{\text{lead}}$ and $p_{\text{lead}}$

Using Matlab, can check:

at $\omega = 2$, $\phi \approx -160^\circ$

— so $\text{PM} = 20^\circ$

(in fact, choosing $K = 4$ made things worse: it increased $\omega_c$ and consequently decreased PM)

We need at least $40^\circ$ phase lead!!

The choice of lead pole/zero must satisfy

$$\sqrt{z_{\text{lead}} \cdot p_{\text{lead}}} \approx 2 \implies z_{\text{lead}} \cdot p_{\text{lead}} = 4$$
Lead & Lag Compensation

Need at least $40^\circ$ phase lead, while satisfying

$$\sqrt{z_{\text{lead}} \cdot p_{\text{lead}}} \approx 2 \implies z_{\text{lead}} \cdot p_{\text{lead}} = 4$$

Let’s try $z_{\text{lead}} = 1$ and $p_{\text{lead}} = 4$

$$D(s) = \frac{s + 1}{s + \frac{1}{4}}$$

Phase lead = $37^\circ$ — not enough!!
Lead & Lag Compensation

Need at least 40° phase lead, while satisfying

$$\sqrt{z_{\text{lead}} \cdot p_{\text{lead}}} \approx 2 \implies z_{\text{lead}} \cdot p_{\text{lead}} = 4$$

The choice of $z_{\text{lead}} = 1$, $p_{\text{lead}} = 4$ gave phase lead = 37°.

Need to space $z_{\text{lead}}$ and $p_{\text{lead}}$ farther apart:

$$\begin{cases} 
z_{\text{lead}} = 0.8, \\
p_{\text{lead}} = 5 \end{cases} \implies \text{phase lead} = 46°$$
Step 3. Evaluate steady-state tracking and choose $z_{\text{lag}}, p_{\text{lag}}$ to satisfy specs

So far:

$$K D(s) G(s) = \begin{cases} \text{lead} & \frac{s}{0.8} + 1 \\ \text{only} & \frac{s}{5} + 1 \end{cases} \cdot \frac{10}{\left(\frac{s}{0.2} + 1\right) \left(\frac{s}{0.5} + 1\right)}$$

$$KD(0)G(0) = 40 \quad \implies \quad e(\infty) = \frac{1}{1 + KD(0)G(0)} = \frac{1}{1 + 40}$$

— this is not small enough: need $1\% = \frac{1}{100} = \frac{1}{1 + 99}$

We want $D(0) \geq \frac{99}{40}$ with lag $\frac{z_{\text{lag}}}{p_{\text{lag}}} \approx 2.5$ will do
Lead & Lag Compensation

Need to choose lag pole/zero that are sufficiently small (not to distort the phase lead too much) and satisfy \( \frac{z_{\text{lag}}}{p_{\text{lag}}} \approx 2.5 \).

We can stick with our previous design:

\[ z_{\text{lag}} = 0.05, \quad p_{\text{lag}} = 0.02 \]

Overall controller:

\[
\frac{s}{4 \cdot \frac{0.8}{s} + 1} \cdot \frac{s + 0.05}{s + 0.02}
\]

(Note: we don’t rewrite lag in Bode form, because \( \frac{z_{\text{lag}}}{p_{\text{lag}}} \) is not incorporated into \( K \).)
Frequency Domain Design Method: Advantages

Design based on Bode plots is good for:

- easily visualizing the concepts
- evaluating the design and seeing which way to change it
- using experimental data (frequency response of the uncontrolled system can be measured experimentally)
Frequency Domain Design Method: Disadvantages

Design based on Bode plots is not good for:

- exact closed-loop pole placement (root locus is more suitable for that)
- deciding if a given $K$ is stabilizing or not ...
  - we can only measure how far we are from instability (using GM or PM), if we know that we are stable
  - however, we don’t have a way of checking whether a given $K$ is stabilizing from frequency response data

What we want is a frequency-domain substitute for the Routh–Hurwitz criterion — this is the Nyquist criterion, which we will discuss in the next lecture.