Plan of the Lecture

- **Review**: stability from frequency response
- **Today’s topic**: control design using frequency response

*Goal*: understand the effect of various types of controllers (PD/lead, PI/lag) on the closed-loop performance by reading the open-loop Bode plot; develop frequency-response techniques for shaping transient and steady-state response using dynamic compensation

*Reading*: FPE, Chapter 6
Review: Phase Margin for 2nd-Order System

\[ G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s}, \quad \text{closed-loop t.f.} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]

\[ \left. \text{PM} \right|_{K=1} = \tan^{-1} \left( \frac{2\zeta}{\sqrt{4\zeta^4 + 1 - 2\zeta^2}} \right) \approx 100 \cdot \zeta \]

Conclusions:

larger PM \iff better damping
(open-loop quantity) \iff (closed-loop characteristic)

Thus, the overshoot \( M_p = \exp \left( -\frac{\pi \zeta}{\sqrt{1-\zeta^2}} \right) \) and resonant peak \( M_r = \frac{1}{2\zeta \sqrt{1-\zeta^2}} - 1 \) are both related to PM through \( \zeta \)!!

\[ \omega_{BW} = \omega_n \sqrt{-s} \quad \Rightarrow \quad \omega_{BW} = \text{closed-loop bandwidth} \]
Assuming that $G(s)$ is minimum-phase (i.e., has no RHP zeros), we derived the following for the Bode plot of $KG(s)$:

<table>
<thead>
<tr>
<th></th>
<th>low freq.</th>
<th>real zero/pole</th>
<th>complex zero/pole</th>
</tr>
</thead>
<tbody>
<tr>
<td>mag. slope</td>
<td>$n$</td>
<td>up/down by 1</td>
<td>up/down by 2</td>
</tr>
<tr>
<td>phase</td>
<td>$n \times 90^\circ$</td>
<td>up/down by $90^\circ$</td>
<td>up/down by $180^\circ$</td>
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</table>

We can state this succinctly as follows:

**Gain-Phase Relationship.** Far enough from break-points,

\[
\text{Phase} \approx \text{Magnitude Slope} \times 90^\circ
\]
Bode’s Gain-Phase Relationship

Gain-Phase Relationship. Far enough from break-points,

$$\text{Phase} \approx \text{Magnitude Slope} \times 90^\circ$$

This suggests the following rule of thumb:

- **$M$** has slope $-2$ at $\omega_c$
  $$\Rightarrow \phi(\omega_c) = -180^\circ$$
  $$\Rightarrow \text{bad (no PM)}$$
- **$M$** has slope $-1$ at $\omega_c$
  $$\Rightarrow \phi(\omega_c) = -90^\circ$$
  $$\Rightarrow \text{good (PM = 90}^\circ)$$

— this is an important *design guideline*!!

(Similar considerations apply when $M$-plot has positive slope – depends on the t.f.)
Gain-Phase Relationship & Bandwidth

M-plot for open-loop t.f. $KG$:

![M-plot](image)

Closed-loop t.f.:

$$T(j\omega_c) = \frac{KG(j\omega_c)}{1 + KG(j\omega_c)} = \frac{-j}{1 - j}$$

$$|T(j\omega)| = \left| \frac{-j}{1 - j} \right| = \frac{1}{\sqrt{2}}$$

$$|T(0)| = \lim_{\omega \to 0} \frac{|KG(j\omega)|}{|1 + KG(j\omega)|} = 1$$

Note: $|KG(j\omega)| \to \infty$ as $\omega \to 0$

If $PM = 90^\circ$, then $\omega_c = \omega_{BW}$

If $PM < 90^\circ$, then $\omega_c \leq \omega_{BW} \leq 2\omega_c$ (see FPE)

$\omega_{BW}$ is by definition freq where CL has phase of $\frac{\pi}{2}$. $\omega_{BW} \geq \omega_n$ (see FPE) $\Delta \leq 2\omega_n$
Bode’s Gain-Phase Relationship suggests that we can shape the time response of the \textit{closed-loop} system by choosing $K$ (or, more generally, a dynamic controller $KD(s)$) to tune the Phase Margin.

In particular, from the quantitative Gain-Phase Relationship,

\[
\text{Magnitude slope}(\omega_c) = -1 \quad \Rightarrow \quad \text{Phase}(\omega_c) \approx -90^\circ
\]

— which gives us PM of 90° and consequently \textbf{good damping}.

$\omega_c \approx \omega_B W$
Example  

Let \( G(s) = \frac{1}{s^2} \) (double integrator)  

Objective: design a controller \( KD(s) \) \((K = \text{scalar gain})\) to give  

- stability \( \times \)  
- good damping (will make this more precise in a bit)  
- \( \omega_{BW} \approx 0.5 \) (always a closed-loop characteristic)  

Strategy:  

- from Bode’s Gain-Phase Relationship, we want magnitude slope = \(-1\) at \( \omega_c \) \( \rightarrow \) \( PM = 90^\circ \) \( \rightarrow \) good damping;  
- if \( PM = 90^\circ \), then \( \omega_c = \omega_{BW} \) \( \rightarrow \) want \( \omega_c \approx 0.5 \)
Design, First Attempt

Let’s try proportional feedback:

\[ D(s) = 1 \implies KD(s)G(s) = KG(s) = \frac{K}{s^2} \]

This is not a good idea: slope = -2 everywhere, so no PM.

We already know that P-gain alone won’t do the job:

\[ K + s^2 = 0 \text{ (imag. poles)} \]
Design, Second Attempt

\[ G(s) + R KD(s) = 1 \]

Let’s try proportional-derivative feedback:

\[ KD(s) = K(\tau s + 1), \quad \text{where } K = K_P, \ K\tau = K_D \]

Open-loop transfer function: \( KD(s)G(s) = \frac{K(\tau s + 1)}{s^2} \).

Bode plot interpretation: PD controller introduces a Type 2 term in the numerator, which pushes the slope up by 1 — this has the effect of pushing the M-slope of \( KD(s)G(s) \) from \(-2\) to \(-1\) past the break-point \( (\omega = 1/\tau) \).
Design, Second Attempt (PD-Control)

Open-loop transfer function: \( KD(s)G(s) = \frac{K(\tau s + 1)}{s^2} \)

For the G-P relationship to be valid, choose the break-point several times smaller than desired \( \omega_c \):

\[ \Rightarrow \text{let’s take } \tau = 10 \]
\[ \Rightarrow \frac{1}{\tau} = 0.1 = \frac{\omega_c}{5} \]

Open-loop t.f.:
Design, Second Attempt (PD-Control)

\[ G(s) + R KD(s) \]

Open-loop transfer function: \( KD(s)G(s) = \frac{K(10s + 1)}{s^2} \)

- Want \( \omega_c \approx 0.5 \)
- This means that
  \[ M(j0.5) = 1 \text{ set } M = 1 \]
  \[ |KD(j0.5)G(j.05)| = \frac{K|5j + 1|}{0.5^2} = 4K \sqrt{26} \approx 20K \]

\[ \implies K = \frac{1}{20} \]
PD Control Design: Evaluation

\[
G(s) = \frac{1}{s^2}
\]

Initial design: \( KD(s) = \frac{10s + 1}{20} \)

What have we accomplished?

- \( PM \approx 90^\circ \) at \( \omega_c = 0.5 \) \( \Rightarrow \omega_{BW} = \omega_c \).
- still need to check in Matlab and iterate if necessary

Trade-offs:

- want \( \omega_{BW} \) to be large enough for fast response (larger \( \omega_{BW} \rightarrow \) larger \( \omega_n \rightarrow \) smaller \( t_r \)), but not too large to avoid noise amplification at high frequencies
- PD control increases slope \( \rightarrow \) increases \( \omega_c \rightarrow \) increases \( \omega_{BW} \rightarrow \) faster response
- usual complaint: D-gain is not physically realizable, so let’s try lead compensation
Lead Compensation: Bode Plot

\[ KD(s) = K \frac{s + z}{s + p}, \quad p \gg z \]

In Bode form:

\[ KD(s) = \frac{Kz(s/z + 1)}{p(\frac{s}{p} + 1)} \]

or, absorbing \( z/p \) into the overall gain, we have

\[ KD(s) = K \frac{(s/z + 1)}{(\frac{s}{p} + 1)} \]

Break-points:

- Type 1 zero with break-point at \( \omega = z \) (comes first, \( z \ll p \))
- Type 1 pole with break-point at \( \omega = p \)
Lead Compensation: Bode Plot

\[ KD(s) = \frac{K \left( \frac{s}{z} + 1 \right)}{\left( \frac{s}{p} + 1 \right)} \]

- magnitude levels off at high frequencies \( \Rightarrow \) better noise suppression
- adds phase, hence the term “phase lead”
Lead Compensation and Phase Margin

\[ KD(s) = \frac{K \left( \frac{s}{z} + 1 \right)}{\left( \frac{s}{p} + 1 \right)} \]

For best effect on PM, \( \omega_c \) should be halfway between \( z \) and \( p \) (on log scale):

\[ \log \omega_c = \frac{\log z + \log p}{2} \]

or \( \omega_c = \sqrt{z \cdot p} \)

— geometric mean of \( z \) and \( p \)

Trade-offs: large \( p - z \) means

- large PM (closer to 90°)
- but also bigger \( M \) at higher frequencies (worse noise suppression)
Back to Our Example: \( G(s) = \frac{1}{s^2} \)

Objectives (same as before):

- stability
- good damping
- \( \omega_{BW} \) close to 0.5

\[ KG(s) = \frac{K}{s^2} \text{ (w/o lead)}: \]

\[ \frac{K}{(0.5)^2} = 1 \implies K = \frac{1}{4} \]

\[ \omega_{BW} = \frac{1}{2} : \quad \omega_{BW} = \omega_c \]

\( \sim \) requires \( \omega_c = \frac{1}{2} \Rightarrow K = \frac{1}{4} \)

after adding lead:

— adding lead will increase \( \omega_c \)!!
Back to Our Example: \( G(s) = \frac{1}{s^2} \)

After adding lead with \( K = \frac{1}{4} \), what do we see?
  
  - adding lead increases \( \omega_c \)
  - \( \Rightarrow \text{PM} < 90^\circ \)
  - \( \Rightarrow \omega_{BW} \text{ may be } > \omega_c \)

To be on the safe side, we choose a new value of \( K \) so that

\[
\omega_c = \frac{\omega_{BW}}{2}
\]

(b/c generally \( \omega_c \leq \omega_{BW} \leq 2\omega_c \))

Thus, we want

\[
\omega_c = 0.25 \quad \Rightarrow \quad K = \frac{1}{16}
\]
Back to Our Example: $G(s) = \frac{1}{s^2}$

Next, we pick $z$ and $p$ so that $\omega_c$ is approximately their geometric mean:

e.g., $z = 0.1$, $p = 2$

$$\sqrt{z \cdot p} = \sqrt{0.2} \approx 0.447$$

Resulting lead controller:

$$KD(s) = \frac{1}{\frac{s}{0.1} + 1}$$

$$= \frac{1}{\frac{s}{2} + 1}$$

(may still need to be refined using Matlab)
Lead Controller Design Using Frequency Response

General Procedure

1. Choose $K$ to get desired bandwidth spec w/o lead
2. Choose lead zero and pole to get desired PM
   ▶ in general, we should first check PM with the $K$ from 1, w/o lead, to see how much more PM we need
3. Check design and iterate until specs are met.

This is an intuitive procedure, but it’s not very precise, requires trial & error.