Plan of the Lecture

- **Review**: basic properties and benefits of feedback control
- **Today’s topic**: introduction to Proportional-Integral-Derivative (PID) control

**Goal**: study basic features and capabilities of PID control (industry standard since 1950’s): arbitrary pole placement; reference tracking; disturbance rejection

**Reading**: FPE, Sections 4.1–4.3; lab manual
Recap: Benefits of Feedback Control

From last lecture: feedback control

- reduces steady-state error to disturbances
- reduces steady-state sensitivity to model uncertainty (parameter variations)
- improves time response

So far, we have only looked at proportional feedback (scalar gain) and 1st-order plants. Now we will add two more basic ingredients and examine their effect on higher-order systems.

We will consider the following plant transfer function:

\[ G(s) = \frac{1}{s^2 - 1} \]

- unstable: poles at \( s = \pm 1 \) (one pole in RHP)
- 2nd-order

- not as easy as DC motor, which was 1st-order and stable.
Proportional Feedback

\[ R \rightarrow \bigoplus E \xrightarrow{K_P} U \xrightarrow{\frac{1}{s^2 - 1}} Y \]

\( K_P \) – “proportional gain” (P-gain) \( \quad U = K_P E \)

Let’s try to find a value of \( K_P \) that would stabilize the system:

\[
\frac{Y}{R} = \frac{K_P}{s^2 - 1} = \frac{K_P}{1 + \frac{K_P}{s^2 - 1}} = \frac{K_P}{s^2 - 1 + K_P}
\]

— the polynomial in the denominator has zero coefficient of \( s \) \( \implies \) necessary condition for stability is not satisfied.

The feedback system is \textit{not stable for any value of} \( K_P \)!!
Derivative Feedback

Let’s feed the derivative of the error, multiplied by some gain, back into the plant:

\[ R \overset{+}{\rightarrow} E \overset{K_D s}{\rightarrow} U \overset{1}{\rightarrow} Y \]

\[ + \quad \frac{1}{s^2 - 1} \quad \]

Motivation: derivative = rate of change; faster change \( \Rightarrow \) more control needed.

Caveat: multiplication by \( s \) is not a causal element (why?)

Derivative action and lack of causality: recall

\[ \dot{e}(t) \approx \frac{e(t + \delta) - e(t)}{\delta} \quad \text{(for small } \delta) \]

— if \( \delta > 0 \), \( e(t + \delta) \) is in the future of \( e(t) \)!!
Disclaimer 1 about D-Feedback: Lack of Causality

Consider some state-space models:

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}
\]

\[
\begin{align*}
sX &= AX + BU \\
y &= CX
\end{align*}
\]

\[
\begin{align*}
(s - A)X &= BU \\
Y &= CB \\
\frac{s - A}{U} &= \frac{CB}{p(s)}
\end{align*}
\]

\[
\deg(q) < \deg(p) \quad \text{— strictly proper transfer function}
\]

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

\[
\begin{align*}
sX &= AX + BU \\
y &= CX + DU
\end{align*}
\]

\[
\begin{align*}
Y &= \frac{CB}{s - A}U + DU \\
&= \frac{CB + D(s - A)}{s - A}U \equiv \frac{q(s)}{p(s)}
\end{align*}
\]

\[
\deg(q) = \deg(p) \quad \text{— proper transfer function}
\]

Causal systems have proper transfer functions.
Lack of Causality

But if \( u = K \dot{e} \), then \( U = KsE \implies \frac{U}{E} = Ks = \frac{q(s)}{p(s)} \)

\( \text{deg}(q) > \text{deg}(p) \) — improper system (lack of causality)

So, \( E \mapsto K_D sE \) is not implementable directly, but we can implement an approximation, e.g.

\[
\frac{K_D as}{a + s} \rightarrow K_D s \quad \text{as} \ a \rightarrow \infty
\]

(this can be done using op-amps).

Alternatively, we can approximate derivative action using finite differences:

\[
\dot{e}(t) \approx \frac{e(t + \delta) - e(t)}{\delta},
\]

but then we must tolerate delays — must wait until time \( t + \delta \) to issue a control signal meant for time \( t \).
Disclaimer 2 about D-Feedback: Noise Amplification

Differentiators amplify noise (noise $\rightarrow$ rapid changes in the reference).

In the lab, D-feedback is implemented differently, in the feedback path. This way, we avoid differentiating the reference, which may be rapidly changing:

$$\frac{Y}{R} = \frac{K_D s G(s)}{1 + K_D s G(s)}$$

Before:

$$\frac{Y}{R} = \frac{K_D s G(s)}{1 + K_D s G(s)}$$

Now:

$$\frac{Y}{R} = \frac{G(s)}{1 + K_D s G(s)}$$

Poles: $1 + K_D s G(s) = 0$

— same poles, but different zeros.

Now the reference signal is smoothed out by the plant $G(s)$ before entering the differentiator, which minimizes distortion due to noise.
Back to Analysis: Derivative Feedback

\[ Y \frac{R}{R} = \frac{K_D s}{s^2 - 1} = \frac{1}{s^2 + K_D s - 1} \]

— still not good: the denominator has a negative coefficient

\[ \Rightarrow \text{not stable; also we have picked up a zero at the origin.} \]

But:

- P-control affected the coefficient of \( s^0 \) (constant term)
- D-control affected the coefficient of \( s \)

— let’s combine them!!
Proportional-Derivative (PD) Control

\[
\frac{Y}{R} = \frac{K_P + K_D s}{s^2 - 1} = \frac{K_P + K_D s}{1 + \frac{K_P + K_D s}{s^2 - 1}}
\]

— now, if we set \( K_D > 0 \) and \( K_P > 1 \), then the transfer function will be stable.

**Even more:** by choosing \( K_P \) and \( K_D \), we can *arbitrarily* assign coefficients of the denominator, and therefore the poles of the transfer function:

PD control gives us *arbitrary pole placement*!!
Proportional-Derivative (PD) Control

\[ Y = \frac{K_P + K_D s}{s^2 + K_D s + K_P - 1} \]

By choosing \(K_P, K_D\), we can achieve arbitrary pole placement!!

Also note that the addition of P-gain moves the zero:

\[ K_D s + K_P = 0 \quad \text{LHP zero at} \quad - \frac{K_P}{K_D} \]

But what’s missing? \( \text{DC gain} = \left. \frac{Y}{R} \right|_{s=0} = \frac{K_P}{K_P - 1} \neq 1 \)

— can’t have perfect tracking of constant reference.
Proportional-Integral-Derivative (PID) Control

Let us try

\[ U = \left( K_P + K_D s + \frac{K_I}{s} \right) E \]  

- the classic three-term controller

In fact, let’s also throw in a constant disturbance:

We will see that, with PID control, the goals of

- tracking a constant reference \( r \)
- rejecting a constant disturbance \( w \)

can be accomplished in one shot.
Proportional-Integral-Derivative (PID) Control

\[ Y = \frac{1}{s^2 - 1} (U + W), \quad U = \left( K_P + K_D s + \frac{K_I}{s} \right) (R - Y) \]

so
\[ Y = \frac{K_P + K_D s + \frac{K_I}{s}}{s^2 - 1} (R - Y) + \frac{1}{s^2 - 1} W \]

Simplify:
\[ (s^2 - 1)Y = \left( K_P + K_D s + \frac{K_I}{s} \right) (R - Y) + W \]
\[ \left( s^2 - 1 + K_P + K_D s + \frac{K_I}{s} \right) Y = \left( K_P + K_D s + \frac{K_I}{s} \right) R + W \]
\[ (s^3 - s + K_P s + K_D s^2 + K_I)Y = (K_P s + K_D s^2 + K_I) R + W s \]
Proportional-Integral-Derivative (PID) Control

\[ Y = \frac{K_Ds^2 + K_PS + K_I}{s^3 + K_Ds^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_Ds^2 + (K_P - 1)s + K_I} W \]
Proportional-Integral-Derivative (PID) Control

\[ Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{1}{s^2 - 1} \]

\[ + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W \]

Stability:

- need \( K_D > 0, \, K_P > 1, \, K_I > 0 \) (necessary condition) and \( K_D(K_P - 1) > K_I \) (Routh–Hurwitz for 3rd-order)
- can still assign coefficients arbitrarily by choosing \( K_P, \, K_I, \, K_D \)
Proportional-Integral-Derivative (PID) Control

Reference tracking:

\[ Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + (K_P - 1) s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1) s + K_I} W \]

\[ \text{DC gain}(R \rightarrow Y) = \frac{K_D s^2 + K_P s + K_I}{s^3 + (K_P - 1) s + K_D s^2 + K_I} \bigg|_{s=0} = 1 \]

— so, with the addition of I-feedback, we remove earlier limitation and achieve perfect tracking!
Proportional-Integral-Derivative (PID) Control

\[ Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W \]

Disturbance rejection:

\[ \text{DC gain}(W \rightarrow Y) = \left. \frac{s}{s^3 + (K_P - 1)s + K_D s^2 + K_I} \right|_{s=0} = 0 \]

— so, integral gain also gives complete attenuation of constant disturbances!!
When the actuator saturates, the error continues to be integrated, resulting in large overshoot.

We say that the integrator “winds up:” the error may be small, but its integrated past history builds up.

There are various anti-windup schemes to deal with this practically important issue. (Essentially, we attempt to detect the onset of saturation and turn the integrator off.)
The fact that $1/s$ leads to perfect tracking of constant references and perfect rejection of constant disturbances is a special case of a more general analysis.

Consider the reference $r(t) = \frac{t^k}{k!}1(t) \iff R(s) = \frac{1}{s^{k+1}}$

Error signal: $E = \frac{1}{1 + KP}R = \frac{1}{1 + KP} \frac{1}{s^{k+1}}$

FVT gives (assuming stability):

$$e(\infty) = sE(s) \bigg|_{s=0} = \frac{1}{1 + KP} \frac{1}{1 + KP s^k} \bigg|_{s=0}$$

— let’s see how the forward gain affects tracking performance.
System Type

System type: the number $n$ of poles the forward-loop transfer function $KP$ has at the origin. It is the degree of the lowest-degree polynomial that cannot be tracked in feedback with zero steady-state error.

Note: the system type is calculated from the forward-loop transfer function, although the conclusions we will draw are about the closed-loop system.
System Type

\[ R(s) = \frac{1}{s^{k+1}} \implies E = \frac{1}{1 + KP} R = \frac{1}{1 + KP} \frac{1}{s^{k+1}} \]

\[ e(\infty) = sE(s) \bigg|_{s=0} = \frac{1}{1 + KP} \frac{1}{s^k} \bigg|_{s=0} \]

— let’s see how forward gain \( KP \) affects tracking performance.

Let’s suppose that \( KP \) has \( n \)th-order pole at \( s = 0 \): \( KP = \frac{K_0}{s^n} \)

\[ sE(s) = \frac{1}{\left(1 + \frac{K_0}{s^n}\right)s^k} = \frac{s^{n-k}}{s^n + K_0} \] — what about \( sE(s) \bigg|_{s=0} ? \)
Let’s suppose that $KP$ has $n$th-order pole at $s = 0$: $KP = \frac{K_0}{s^n}$

$$sE(s) = \frac{1}{(1 + \frac{K_0}{s^n}) s^k} = \frac{s^{n-k}}{s^n + K_0}$$

— what about $sE(s)\bigg|_{s=0}$?

Recall: reference $r(t)$ is a polynomial of degree $k$

Three cases to consider —

- $n > k$: $e(\infty) = 0$  
  perfect tracking

- $n = k$: $e(\infty) = \text{const} \neq 0$  
  imperfect tracking

- $n < k$: $e(\infty) = \infty$  
  no tracking
System type is the degree of the lowest-degree polynomial that cannot be tracked in feedback with zero steady-state error.

- **Type 0**: no pole at the origin. This is what we had without the I-gain: nonzero SS error to constant references.
- **Type 1**: a single pole at the origin. This is what we get with I-gain: can track (respectively, reject) constant references (respectively, disturbances) with zero error.
  - can check that we have a nonzero (but finite) error when tracking ramp references
- **Type 2**: a double pole at the origin. Can track ramp references without error, but not $t^2, t^3, ...$
**PID Control: Summary & Further Comments**

**P-gain** simplest to implement, but not always sufficient for stabilization

**D-gain** helps achieve stability, improves time response (more control over pole locations)
  
  - arbitrary pole placement only valid for 2nd-order response; in general, we still have control over two *dominant poles*
  - cannot be implemented directly, so need approximate implementation; D-gain also amplifies noise

**I-gain** essential for perfect steady-state tracking of constant reference and rejection of constant disturbance
  
  - but $1/s$ is not a stable element by itself, so one must be careful: it can destabilize the system if the feedback loop is broken (integrator wind-up)

*given* keep PE