Plan of the Lecture

Review: stability; Routh–Hurwitz criterion

Today's topic: basic properties and benefits of feedback control

Goal: understand the difference between open-loop and closed-loop (feedback) control; examine the benefits of feedback: reference tracking and disturbance rejection; reduction of sensitivity to parameter variations; improvement of time response.

Reading: FPE, Section 4.1; lab manual
Two Basic Control Architectures

- Open-loop control

- Feedback (closed-loop) control

Here, \( W \) is a *disturbance*; \( K \) is *not necessarily* a static gain
Basic Objectives of Control

- track a given reference
- reject disturbances
- meet performance specs

Intuitively, the difference between the open-loop and the closed-loop architectures is clear (think cruise control ...)

\[ R \rightarrow K \rightarrow P \rightarrow U \rightarrow Y \]

\[ R \rightarrow E \rightarrow K \rightarrow U \rightarrow P \rightarrow Y \]
Open-Loop Control

- cheaper/easier to implement (no sensor required)
- does not destabilize the system

e.g., if both $K$ and $P$ are stable (all poles in OLHP),

\[ \frac{Y}{R} = KP \]

is also stable:

\[ \{\text{poles of } KP\} = \{\text{poles of } K\} \cup \{\text{poles of } P\} \]
Feedback Control

- more difficult/expensive to implement (requires a sensor and an information path from controller to actuator)
- may destabilize the system:

\[
\frac{Y}{R} = \frac{K P}{1 + K P}
\]

has new poles, which may be unstable

- but: feedback control is the *only way* to stabilize an unstable plant (this was the Wright brothers’ key insight)
Benefits of Feedback Control

Feedback control:

- reduces steady-state error to disturbances
- reduces steady-state sensitivity to model uncertainty (parameter variations)
- improves time response
Case Study: DC Motor

Inputs: 
- $v_a$ – input voltage
- $\tau_e$ – load/disturbance torque

Outputs: 
- $\omega_m$ – angular speed of the motor

Transfer function:

$$\Omega_m = \frac{A}{\tau s + 1} V_a + \frac{B}{\tau s + 1} T_e$$

- $\tau$ – time constant
- $A, B$ – system gains

Objective: have $\Omega_m$ approach and track a given reference $\Omega_{ref}$ in spite of disturbance $T_e$. 
Two Control Configurations

- **Open-loop control**
  \[ J_m = \frac{k_{ol} A}{26 + 1} \]
  \[ J_m(\infty) = J_m \text{ref.} \]
  \[ \text{If } K_{ol} = \frac{1}{A}, \text{ then } J_m = \frac{1}{26} J_m \text{ref.} \text{ from FVT} \]

- **Feedback (closed-loop) control**
  \[ \Omega_{\text{ref}} \rightarrow K_{cl} \rightarrow \text{error} \]
  \[ \text{If } K_{cl} = \frac{A}{\tau s + 1} \text{ and } 1 + k_{ol} A/(\tau s + 1) \text{ is good, then} \]
  \[ J_m(\infty) = \frac{4}{A_{f1}} \]
Disturbance Rejection

Goal: maintain $\omega_m = \omega_{\text{ref}}$ in steady state in the presence of constant disturbance.

Open-loop: 

- the controller receives *no information* about the disturbance $\tau_e$ (the only input is $\omega_{\text{ref}}$, no feedback signal from anywhere else)
- so, let’s attempt the following: design for *no disturbance* (i.e., $\tau_e = 0$), then see how the system works in general
Disturbance Rejection: Open-Loop Control

First assume zero disturbance:

\[ \frac{A}{\tau s + 1} \] (stable pole at \( s = -1/\tau \))

We want DC gain = 1

\[ \Omega_m = \frac{A}{\tau s + 1} V_a = \frac{K_{ol} A}{\tau s + 1} \Omega_{ref} \]

Let’s just use constant gain: \( K_{ol} = 1/A \)

\[ \omega_m(\infty) = \frac{1}{A} \cdot A \cdot \omega_{ref} = \omega_{ref} \quad (\text{for } T_e = 0) \]

What happens in the presence of nonzero \( T_e \)?

\[ \Omega_m = \frac{A}{\tau s + 1} \frac{1}{A} \Omega_{ref} + \frac{B}{\tau s + 1} T_e \]

\[ \omega_m(\infty) = \omega_{ref} + B \tau_e \]

step input

step input
Disturbance Rejection: Open-Loop Control

Steady-state motor speed for constant reference and constant disturbance:

$$\omega_m(\infty) = \omega_{\text{ref}} + B\tau_e$$

Conclusion: in the absence of disturbances, reference tracking is good, but disturbance rejection is pretty poor. Steady-state error is determined by $B$, and we have no control over it (and, in fact, cannot change this through any choice of controller $K_{\text{ol}}$, no matter how clever)
Disturbance Rejection: Feedback Control

\[ V_a = K_{cl} E = K_{cl} (\Omega_{\text{ref}} - \Omega_m) \]

\[ \Omega_m = \frac{A}{\tau s + 1} K_{cl} (\Omega_{\text{ref}} - \Omega_m) + \frac{B}{\tau s + 1} T_e \]

Solve for \( \Omega_m \):

\[ (\tau s + 1)\Omega_m = AK_{cl} (\Omega_{\text{ref}} - \Omega_m) + BT_e \]

\[ \Omega_m = \frac{AK_{cl}}{\tau s + 1 + AK_{cl}} \Omega_{\text{ref}} + \frac{B}{\tau s + 1 + AK_{cl}} T_e \]
Disturbance Rejection: Feedback Control

\[ T_e \]

\[ B/A \]

\[ \Omega_m = \frac{AK_{cl}}{\tau s + 1 + AK_{cl}} \Omega_{\text{ref}} + \frac{B}{\tau s + 1 + AK_{cl}} T_e \]

(provided all transfer functions are strictly stable)

Assuming that the reference \( \omega_{\text{ref}} \) and the disturbance \( \tau_e \) are constant, we apply FVT:

\[ \omega_m(\infty) = \frac{AK_{cl}}{1 + AK_{cl}} \omega_{\text{ref}} + \frac{B}{1 + AK_{cl}} \tau_e \]
Disturbance Rejection: Feedback Control

Steady-state speed for constant reference and disturbance:

\[ \omega_m(\infty) = \frac{AK_{\text{cl}}}{1 + AK_{\text{cl}}} \omega_{\text{ref}} + \frac{B}{1 + AK_{\text{cl}}} \tau_e \]

Conclusions:

- \( \frac{AK_{\text{cl}}}{1 + AK_{\text{cl}}} \neq 1 \), but can be brought arbitrarily close to 1 when \( K_{\text{cl}} \to \infty \). Thus, steady-state tracking is good with high gain, but never quite as good as in open-loop case.

- \( \frac{B}{1 + AK_{\text{cl}}} \) is small (arbitrarily close to 0) for large \( K_{\text{cl}} \). Thus, much better disturbance rejection than with open-loop control.
Sensitivity to Parameter Variations

Consider again our DC motor model, with no disturbance:

\[
\begin{align*}
\Omega_{\text{ref}} & \rightarrow K_{\text{ol}} & V_a & \rightarrow \frac{A}{\tau s + 1} & \rightarrow \Omega_m \\
& \text{open-loop controller} & & \text{motor} & \\
\Omega_{\text{ref}} & \rightarrow + & K_{\text{cl}} & \rightarrow \frac{A}{\tau s + 1} & \rightarrow \Omega_m
\end{align*}
\]

Bode’s sensitivity concept: In the “nominal” situation, we have the motor with DC gain = \(A\), and the overall transfer function, either open- or closed-loop, has some other DC gain (call it \(T\)).

Now suppose that, due to modeling error, changes in operating conditions, etc., the motor gain changes:

\[
A \rightarrow A + \delta A
\]

This will cause a perturbation in the overall DC gain:

\[
T \rightarrow T + \delta T \quad \text{(from calculus, to 1st order, } \delta T \approx \frac{dT}{dA} \delta A)\]
Sensitivity to Parameter Variations

\[ A \rightarrow A + \delta A \quad \text{(small perturbation in system gain)} \]
\[ T \rightarrow T + \delta T \quad \text{(resultant perturbation in overall DC gain)} \]

Hendrik Wade Bode
(1905–1982)

Bode’s sensitivity:

\[ S \triangleq \frac{\delta T/T}{\delta A/A} \]

\( S \) = relative error
= \( \frac{\text{normalized (percentage) error in } T}{\text{normalized (percentage) error in } A} \)
Sensitivity to Parameter Variations

Let’s compute $S$ for our DC motor control example, both open- and closed-loop.

Open-loop:
- nominal case
  \[ T_{ol} = K_{ol} A = \frac{1}{A} A = 1 \]  
  \( \leftarrow T_{ol} \text{ is DC current} \)
- perturbed case

\[ A \rightarrow A + \delta A \]

\[ T_{ol} \rightarrow K_{ol} (A + \delta A) = \frac{1}{A} (A + \delta A) = 1 + \frac{\delta A}{A} \]

Sensitivity:
\[ S_{ol} = \frac{\delta T_{ol}}{T_{ol}}/\frac{\delta A_{ol}}{A_{ol}} = \frac{\delta A}{A} = 1 \]

For example, a 5% error in $A$ will cause a 5% error in $T_{ol}$. 
Sensitivity to Parameter Variations

Closed-loop:

- nominal case
  \[ T_{cl} = \frac{AK_{cl}}{1 + AK_{cl}} \]
- perturbed case
  \[ A \rightarrow A + \delta A \quad T_{cl} \rightarrow T_{cl} + \delta T_{cl} \]

Taylor expansion:

\[ T(A + \delta A) = T(A) + \frac{dT}{dA}(A)\delta A + \text{higher-order terms} \]

In our case:

\[ \frac{dT_{cl}}{dA} = \frac{K_{cl}}{1 + AK_{cl}} - \frac{AK_{cl}^2}{(1 + AK_{cl})^2} = \frac{K_{cl}}{(1 + AK_{cl})^2} \]

\[ \delta T_{cl} = \frac{K_{cl}}{(1 + AK_{cl})^2} \delta A \]
Sensitivity to Parameter Variations

From before:

\[ \delta T_{cl} = \frac{K_{cl}}{(1 + AK_{cl})^2} \delta A \]

\[ T_{cl} = \frac{AK_{cl}}{1 + AK_{cl}} \]

Therefore

\[ \frac{\delta T_{cl}}{T_{cl}} = \frac{AK_{cl}}{(1 + AK_{cl})^2} \delta A = \frac{1}{1 + AK_{cl}} \frac{\delta A}{A} \]

Sensitivity: \( S_{cl} = \frac{\delta T_{cl}/T_{cl}}{\delta A/A} = \frac{1}{1 + AK_{cl}} \) \( (\ll 1 \text{ for large } K_{cl}) \)

With high-gain feedback, we get smaller relative error due to parameter variations in the plant model.
Time Response

We still assume no disturbance: $\tau_e = 0$.

So far, we have focused on DC gain only (steady-state response). What about transient response?

Open-loop

$$\Omega_m = \frac{AK_{cl}}{\tau s + 1} \Omega_{ref}$$

Pole at $s = -\frac{1}{\tau}$ $\implies$ transient response is $e^{-t/\tau}$

Here, $\tau$ is the time constant: the time it takes the system response to decay to $1/e$ of its starting value.

In the open-loop case, larger time constant means faster convergence to steady state. This is not affected by the choice of $K_{cl}$ in any way!
Time Response

Closed-loop

\[
\Omega_m = \frac{AK_{cl}}{\tau s + 1 + AK_{cl}} \Omega_{ref}
\]

Closed-loop pole at \( s = -\frac{1}{\tau} (1 + AK_{cl}) \)

(the only way to move poles around is via feedback)

Now the transient response is \( e^{-\frac{1+AK_{cl}}{\tau} t} \), with

\[
\text{time constant} = \frac{\tau}{1 + AK_{cl}}
\]

— for large \( K_{cl} \), we have a much smaller time constant, i.e.,

* faster convergence * to steady-state.
Feedback control:

- reduces steady-state error to disturbances
- reduces steady-state sensitivity to model uncertainty (parameter variations)
- improves time response

Word of caution: high-gain feedback only works well for 1st-order systems; for higher-order systems, it will typically cause underdamping and instability.

Thus, we need a more sophisticated design than just static gain. We will take this up in the next lecture with Proportional-Integral-Derivative (PID) control.