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Reading: FPE, Sections 5.1–5.4, 6.1

Recap: Lead & Lag Compensators

Consider a general controller of the form

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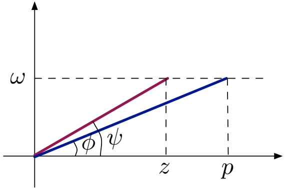
- if $z < p$, then $\psi - \phi > 0$ (phase lead)
- if $z > p$, then $\psi - \phi < 0$ (phase lag)



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We use lag controllers as dynamic compensators for approximate PI control.

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$$s^2 + (K+p-1)s + Kz - p = 0$$

Conditions for stability: $K > 1-p,\ Kz > p$

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Tracking a constant reference: if the stability conditions

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Lag compensation *does not* give perfect tracking (indeed, it does not change system type), but we can get as good a tracking as we want by playing with K, z, p. On the other hand, unlike PI, lag compensation gives a stable controller.

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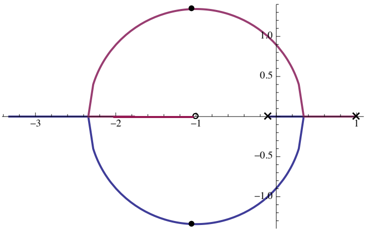
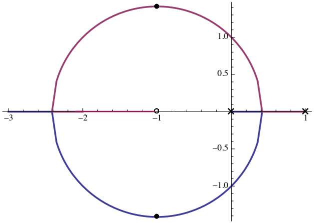


Compare to PI root locus:



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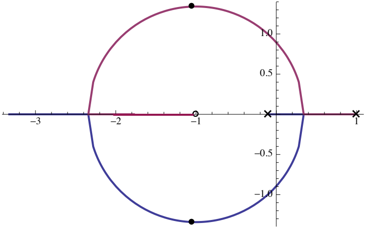

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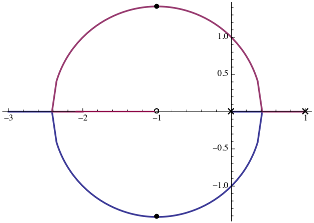

What do we see? Compared to PD vs. lead, there is no qualitative change in the shape of RL, since we are not changing #(poles) or #(zeros).

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Main technique: select parameters to satisfy the phase condition (points on RL must be such that $\angle L(s) = 180°$).

Caveat: may not always be possible!

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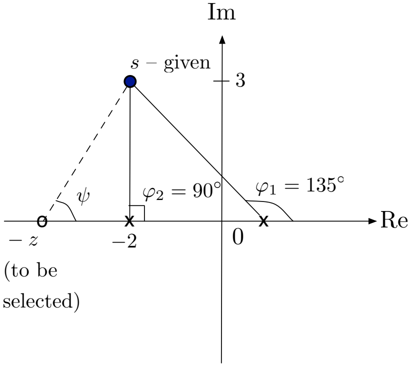

Must have

$$\underbrace{\psi}_{\text{angle from } s \text{ to zero}} - \sum_i \underbrace{\varphi_i}_{\text{angles from } s \text{ to poles}} = 180^\circ$$

So, we want $\psi = 180^\circ + \sum_i \varphi_i$

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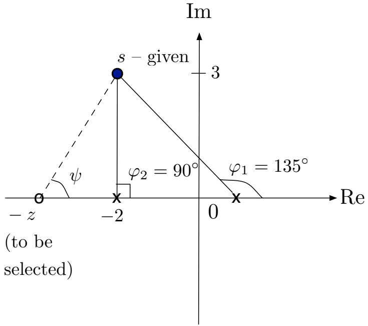

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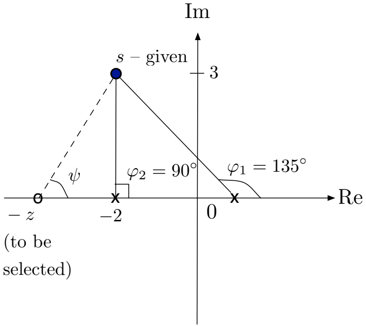

We have

$\varphi_1 = 135°$,

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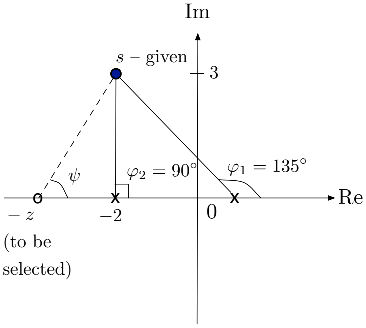

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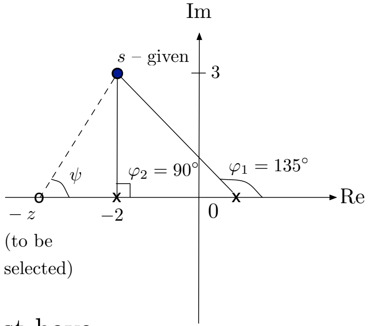

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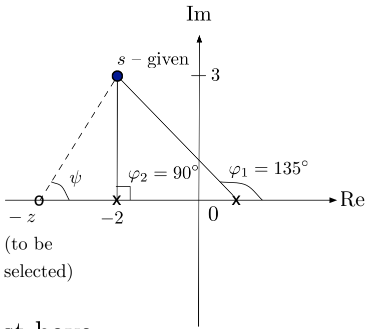

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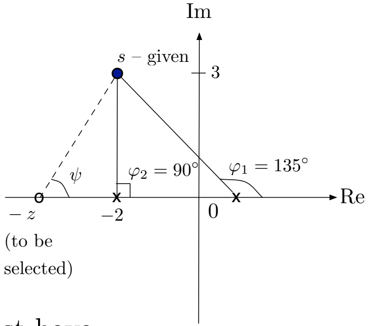

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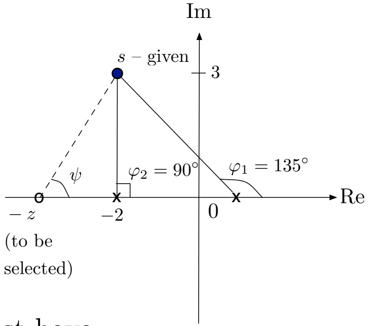

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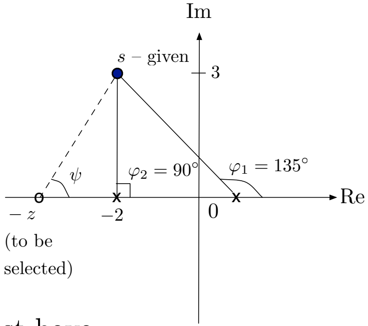

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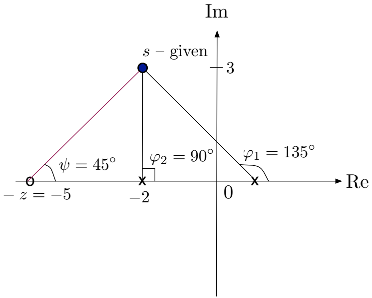

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so we need $K = 3$

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- compare against desired characteristic polynomial:

$$s^2 + 4s + 13 \qquad \Longrightarrow \qquad K+1 = 4, \ 5K - 2 = 13$$

  so we need $K = 3$

- compute s.s. tracking error:

Let $G_p(s) = \dfrac{1}{s-1}, \qquad G_c(s) = K\dfrac{s+z}{s+p}$

Problem: given $p = 2$, find z to place poles at $-2 \pm 3j$.

Solution:

- we already found that we need $z = 5$
- resulting characteristic polynomial:

$$(s-1)(s+2) + K(s+5)$$
$$s^2 + (K+1)s + 5K - 2$$

- compare against desired characteristic polynomial:

$$s^2 + 4s + 13 \qquad \Longrightarrow \qquad K+1 = 4, \; 5K - 2 = 13$$

  so we need $K = 3$

- compute s.s. tracking error: $\left| \dfrac{1}{1 - \frac{Kz}{p}} \right| = \dfrac{1}{6.5} \approx 15\%$

Story So Far

PD control:

Story So Far

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- provides stability, allows to shape transient response specs

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PI control:

- provides stability and perfect steady-state tracking of constant references

Story So Far

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PI control:

- provides stability and perfect steady-state tracking of constant references
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Story So Far

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- provides stability, allows to shape transient response specs
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PI control:

- provides stability and perfect steady-state tracking of constant references
- replace unstable I-controller K/s with a stable lag controller $K\frac{s+z}{s+p}$, where $p < z$
- this does not change the shape of RL compared to PI

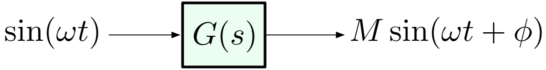
What About PID Control?

Obvious solution — combine lead *and* lag compensation.

We will develop this further in homework and later in the course using frequency-response design methods — which are the subject of several lectures, starting with today's.

The Frequency-Response Design Method

Recall the frequency-response formula:



where $M = M(\omega) = |G(j\omega)|$ and $\phi = \phi(\omega) = \angle G(j\omega)$

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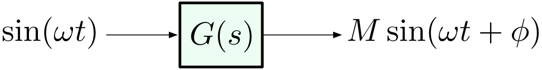

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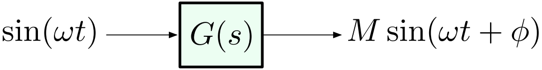

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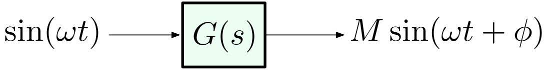

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$$\sin(\omega t) \longmapsto \frac{G(j\omega)e^{j\omega t} - G(-j\omega)e^{-j\omega t}}{2j}$$

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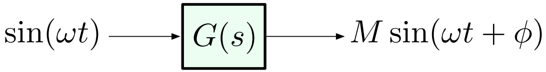

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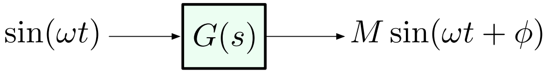

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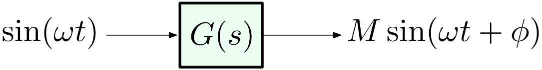

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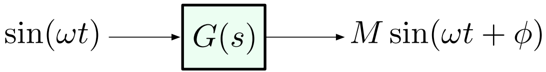
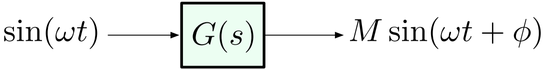


where $M = M(\omega) = |G(j\omega)|$ and $\phi = \phi(\omega) = \angle G(j\omega)$

Let's apply this formula to our prototype 2nd-order system:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The Frequency-Response Design Method



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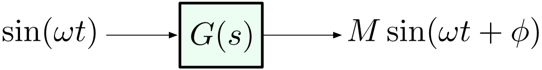
The Frequency-Response Design Method



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$$= \left| \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta\frac{\omega}{\omega_n} j} \right|$$

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$$
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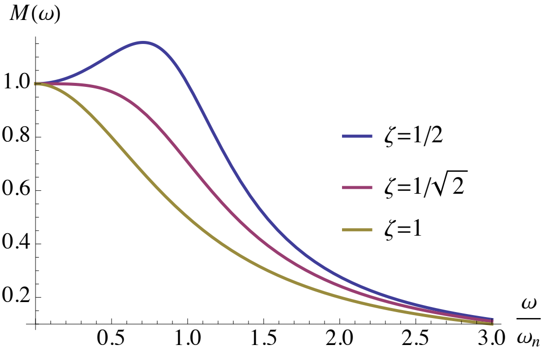
&= \left| \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta\frac{\omega}{\omega_n} j} \right| \\
&= \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}}
\end{aligned}
$$

The Frequency-Response Design Method

For our prototype 2nd-order system:

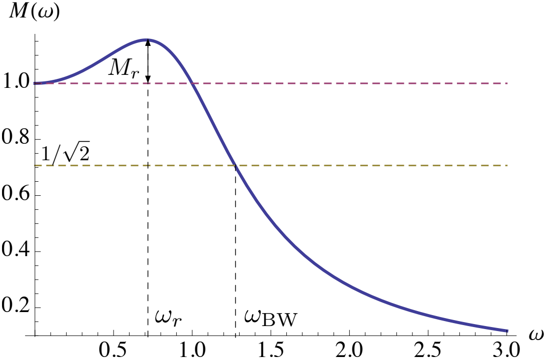
$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$M(\omega) = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}} = \frac{1}{\sqrt{1 + (4\zeta^2 - 2)\left(\frac{\omega}{\omega_n}\right)^2 + \left(\frac{\omega}{\omega_n}\right)^4}}$$



Frequency Response Parameters

Here is a typical frequency response magnitude plot:

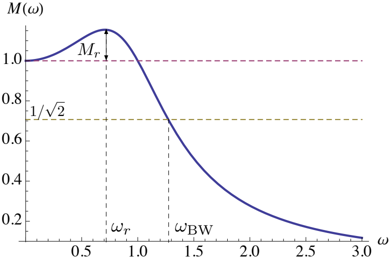


ω_r – resonant frequency

M_r – resonant peak

ω_{BW} – bandwidth

Frequency Response Parameters



We can get the following formulas using calculus:

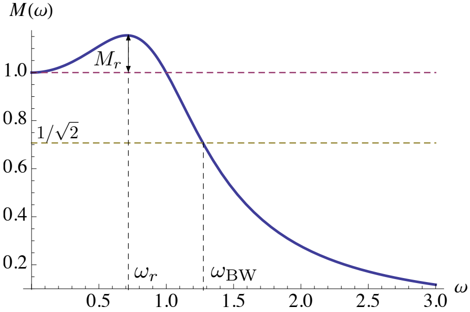
$$\begin{cases} \omega_r = \omega_n \sqrt{1-2\zeta^2} \\ M_r = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}} - 1 \end{cases} \qquad \left(\text{valid for } \zeta < \frac{1}{\sqrt{2}};\ \text{for } \zeta \ge \frac{1}{\sqrt{2}},\ \omega_r = 0\right)$$

$$\omega_{\rm BW} = \omega_n \underbrace{\sqrt{(1-2\zeta^2) + \sqrt{(1-2\zeta^2)^2 + 1}}}_{=1 \text{ for } \zeta = 1/\sqrt{2}}$$

— so, if we know $\omega_r, M_r, \omega_{\rm BW}$, we can determine ω_n, ζ and hence the time-domain specs (t_r, M_p, t_s)

Frequency Response & Time-Domain Specs

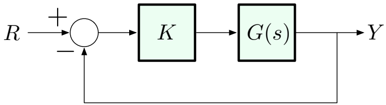
All information about time response is also encoded in frequency response!!



$$\text{small } M_r \longleftrightarrow \text{better damping}$$

$$\text{large } \omega_{\mathrm{BW}} \longleftrightarrow \text{large } \omega_n \longleftrightarrow \text{smaller } t_r$$

Frequency-Response Design Method: Main Idea



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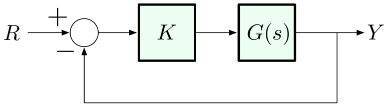
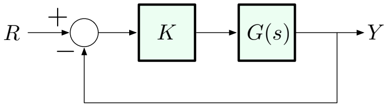


Two-step procedure:

Frequency-Response Design Method: Main Idea



Two-step procedure:

1. Plot the frequency response of the *open-loop* transfer function $KG(s)$ [or, more generally, $D(s)G(s)$], at $s = j\omega$

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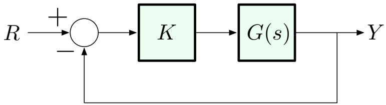

Two-step procedure:

1. Plot the frequency response of the *open-loop* transfer function $KG(s)$ [or, more generally, $D(s)G(s)$], at $s = j\omega$
2. See how to relate this open-loop frequency response to closed-loop behavior.

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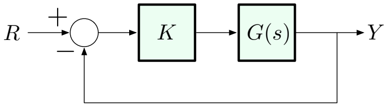

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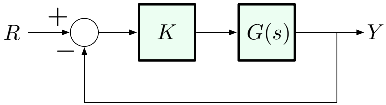

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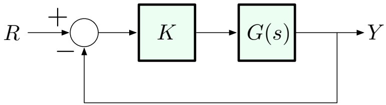

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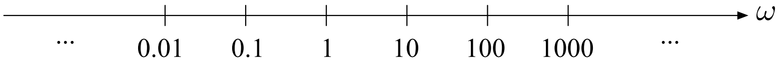
1. Bode plots: magnitude $|KG(j\omega)|$ and phase $\angle KG(j\omega)$ vs. frequency ω (could have seen it earlier, in ECE 342)
2. Nyquist plots: $\text{Im}\big(KG(j\omega)\big)$ vs. $\text{Re}\big(K(j\omega)\big)$ [Cartesian plot in s-plane] as ω ranges from $-\infty$ to $+\infty$

Note on the Scale

Horizontal (ω) axis:

we will use *logarithmic scale* (base 10) in order to display a wide range of frequencies.

Note: we will still mark the values of ω, *not* $\log_{10}\omega$, on the axis, but the *scale* will be logarithmic:



Equal intervals on log scale correspond to decades in frequency.

Note on the Scale

Vertical axis on magnitude plots:

we will also use logarithmic scale, just like the frequency axis.

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we will also use logarithmic scale, just like the frequency axis.

Reason:

$$|(M_1 e^{j\phi_1})(M_2 e^{j\phi_2})| = M_1 \cdot M_2$$
$$\log(M_1 M_2) = \log M_1 + \log M_2$$

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— this means that we can simply *add* the graphs of $\log M_1(\omega)$ and $\log M_2(\omega)$ to obtain the graph of $\log\big(M_1(\omega)M_2(\omega)\big)$, and graphical addition is easy.

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Decibel scale:

$$(M)_{\text{dB}} = 20\log_{10} M \qquad \text{(one decade} = 20\,\text{dB)}$$

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Vertical axis on phase plots:

we will plot the phase on the usual (linear) scale.

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we will plot the phase on the usual (linear) scale.

Reason:

$$\angle\left((M_1 e^{j\phi_1})(M_2 e^{j\phi_2})\right) = \angle\left(M_1 M_2 e^{j(\phi_1+\phi_2)}\right)$$
$$= \phi_1 + \phi_2$$

Note on the Scale

Vertical axis on phase plots:

we will plot the phase on the usual (linear) scale.

Reason:

$$\begin{aligned}\angle\left((M_1 e^{j\phi_1})(M_2 e^{j\phi_2})\right) &= \angle\left(M_1 M_2 e^{j(\phi_1+\phi_2)}\right)\\ &= \phi_1 + \phi_2\end{aligned}$$

— this means that we can simply *add* the phase plots for two transfer functions to obtain the phase plot for their product.

Scale Convention for Bode Plots

	magnitude	phase
horizontal scale	log	log
vertical scale	log	linear

Advantage of the scale convention: we will learn to do Bode plots by starting from simple factors and then building up to general transfer functions by considering products of these simple factors.