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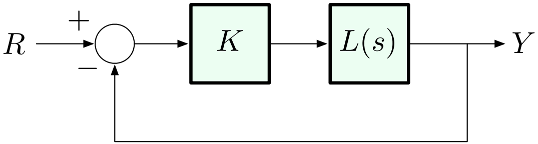
Note!! The way I teach the Root Locus differs a bit from what the textbook does (good news: it is simpler). Still, pay attention in class!!

Reminder: Root Locus



$$\text{where } L(s) = \frac{b(s)}{a(s)} = \frac{s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n},\ m \leq n$$

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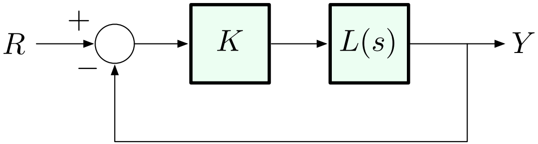

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Root locus: the set of all $s \in \mathbb{C}$ that solve the *characteristic equation*

$$a(s) + Kb(s) = 0$$

as K varies from 0 to ∞.

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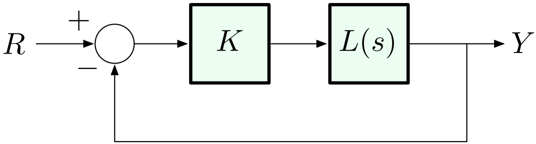

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Or equivalently:

The phase condition: The root locus of $1 + KL(s)$ is the set of all $s \in \mathbb{C}$, such that $\angle L(s) = 180°$, i.e., $L(s)$ is real and negative.

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There are *six rules* for sketching root loci. These rules are mainly qualitative, and their purpose is to give intuition about impact of poles and zeros on performance.

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Last time, we have covered Rules A–C (and a bit of D ...)

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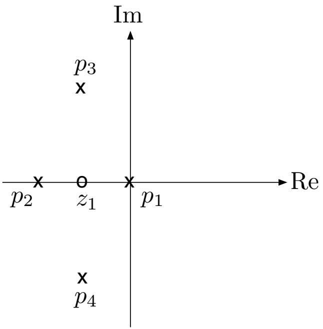

Example, continued

Three more rules:

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Characteristic equation in our example:

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— don't even think about factoring this polynomial!!

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— this sum must be $\pm 180^\circ$ for *any* s that lies on the RL.

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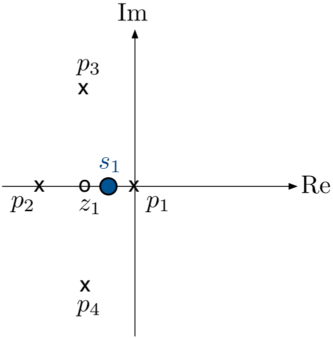
So, we try test points:



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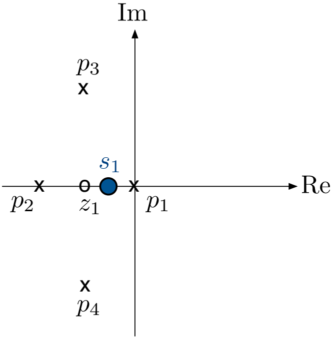

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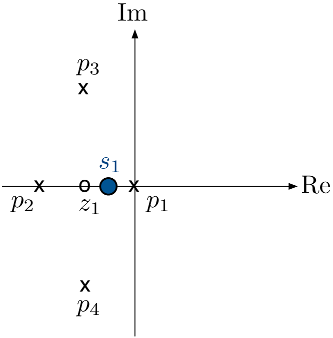

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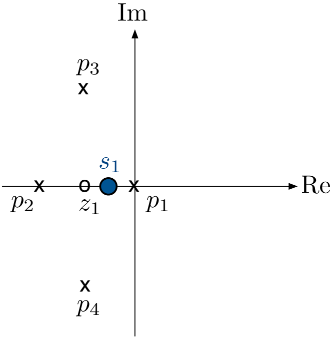

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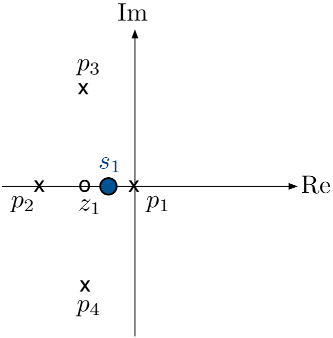

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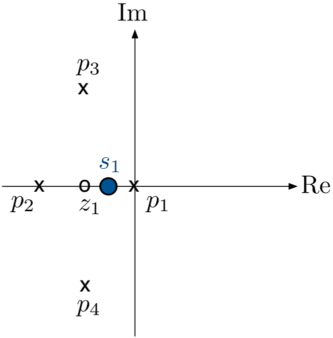

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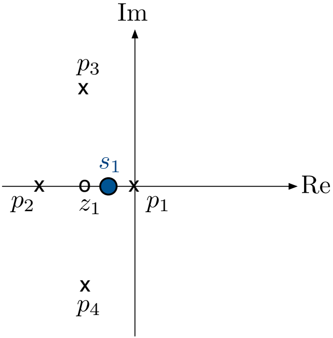

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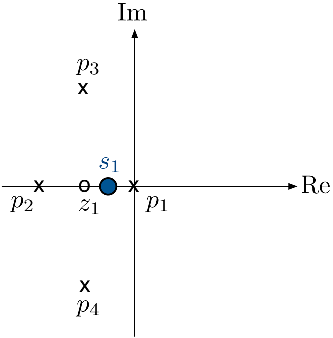

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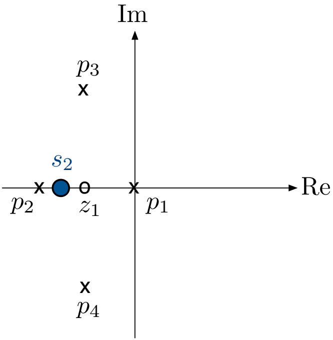

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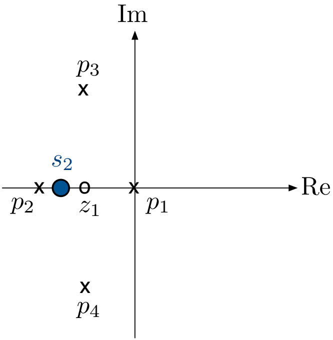

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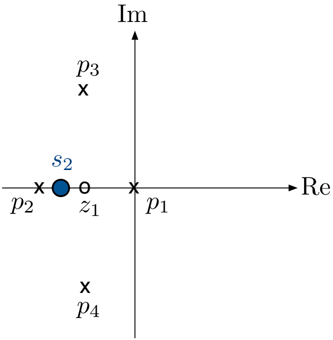

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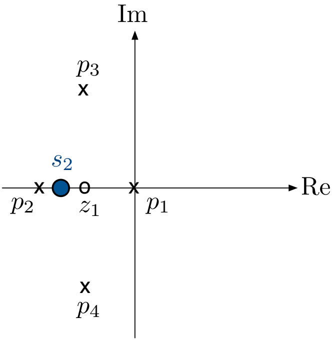

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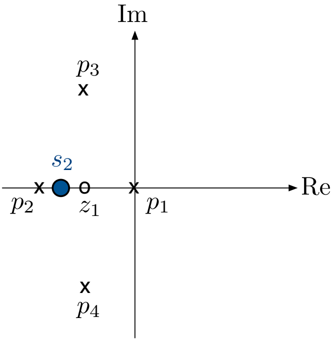

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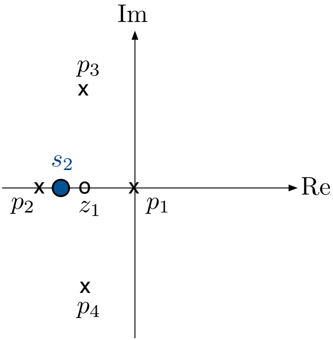

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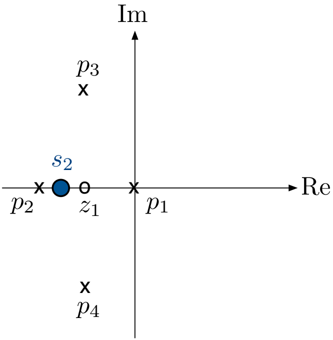

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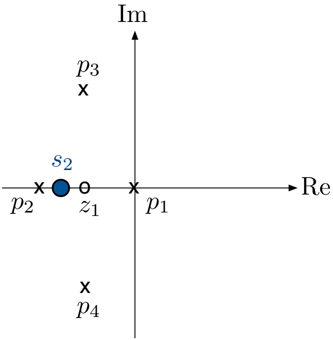

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✕ s_1 is not on RL

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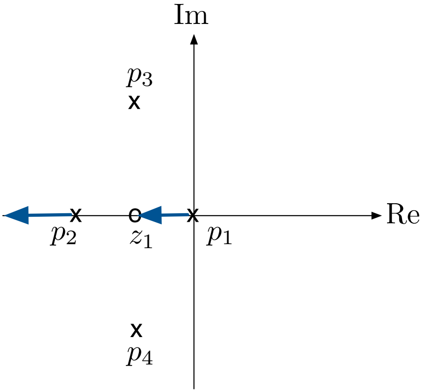

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Rule E: Asymptotes

Rule E: Branches near ∞ have phase

$$\begin{aligned} \angle s &\simeq \frac{180^\circ + \ell \cdot 360^\circ}{n-m} \\ &= \frac{(2\ell+1)\cdot 180^\circ}{n-m}, \qquad \ell = 0, 1, \ldots, n-m-1 \end{aligned}$$

Note: if $m = n$, then there are no branches at ∞.

Back to Example: Rule E

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— asymptotes have angles 60°, 180°, -60°

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Do the branches of the root locus cross the $j\omega$ axis?
(transition from *stability* to *instability*)

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Best approach here: use the *Routh test* to first determine the critical value of K (when the characteristic polynomial becomes unstable), then plug it in and solve for $j\omega$-crossings (numerically or analytically).

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In our example, the characteristic polynomial is

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The characteristic polynomial is stable for $K < \sqrt{80} = 4\sqrt{5}$

$$\Longrightarrow K_{\text{critical}} = 4\sqrt{5}$$

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$j\omega$-crossing at $j\omega_0 = \sqrt{1 + \sqrt{5}} \approx 1.8$, when $K = 4\sqrt{5} \approx 8.9$

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(transition from stability to instability)

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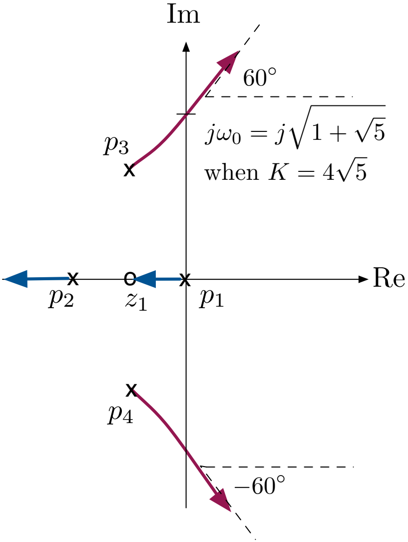

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Control Design Using Root Locus

Case study: double integrator, transfer function $G(s) = \dfrac{1}{s^2}$

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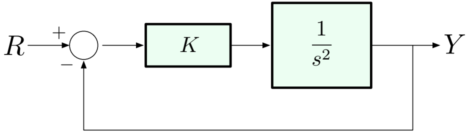
First, let's try a simple P-gain:

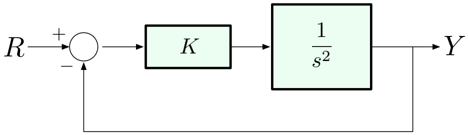


Control Design Using Root Locus

Case study: double integrator, transfer function $G(s) = \dfrac{1}{s^2}$

Control objective: ensure stability; meet time response specs.

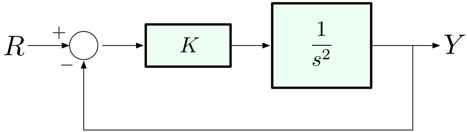
First, let's try a simple P-gain:



Closed-loop transfer function:

$$\frac{\frac{K}{s^2}}{1 + \frac{K}{s^2}} = \frac{K}{s^2 + K}$$

Double Integrator with P-Gain



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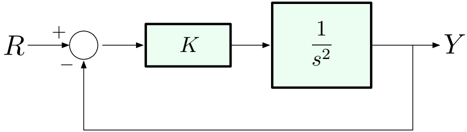

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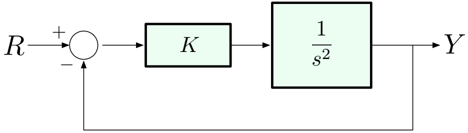

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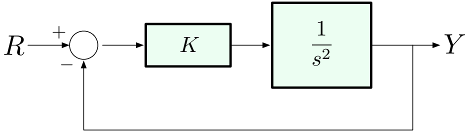

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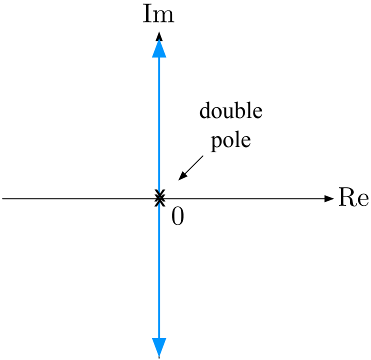

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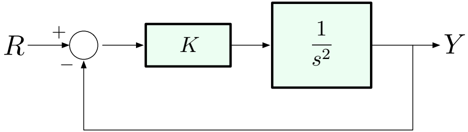

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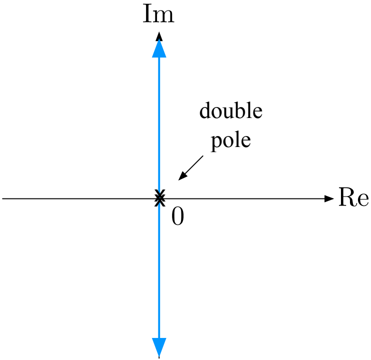
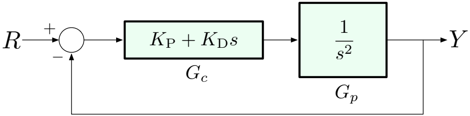


This confirms what we already knew: P-gain alone does not deliver stability.

Double Integrator with PD-Control



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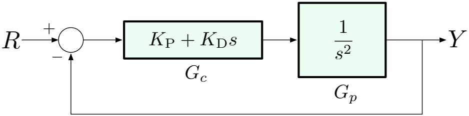

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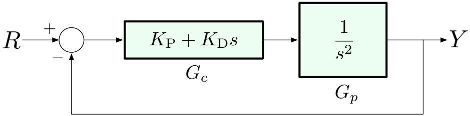

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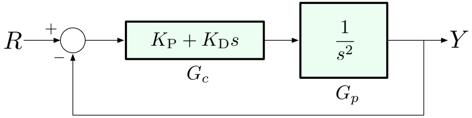

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To use the RL method, we need to convert it into the Evans form $1 + KL(s) = 0$, where $L(s) = \frac{b(s)}{a(s)} = \frac{s^m + b_1 s^{m-1} + \ldots}{s^n + a_1 s^{n-1} + \ldots}$

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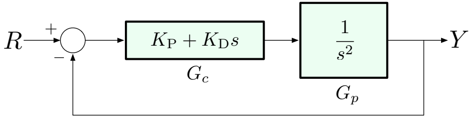

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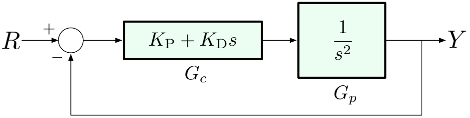

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$$\Longrightarrow \ K = K_D, \ L(s) = \frac{s + K_P/K_D}{s^2} \qquad \text{(assume } K_P/K_D \text{ fixed, } = 1)$$

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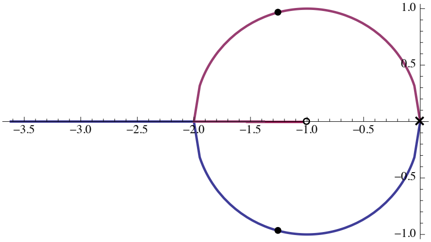

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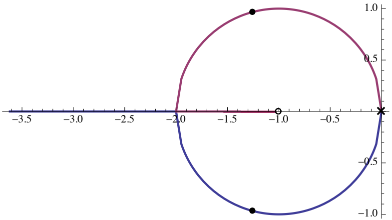

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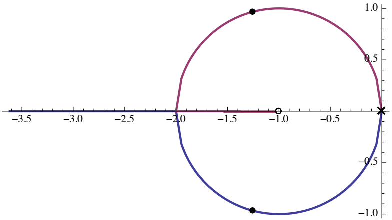

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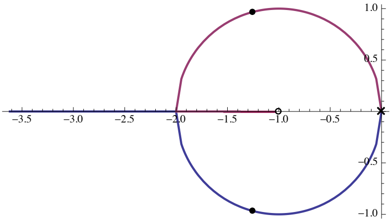

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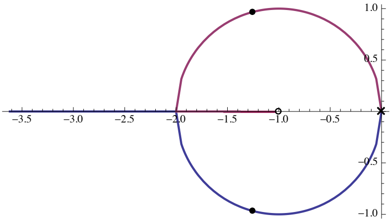

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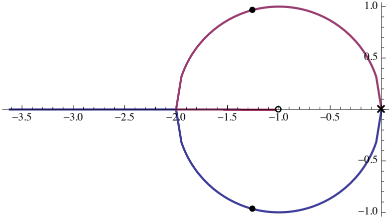

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So, the effect of D-gain was to introduce an *open-loop zero* into LHP, and this zero "pulled" the root locus into LHP, thus stabilizing the system.

Dynamic Compensation

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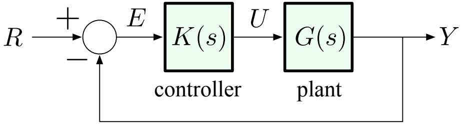

— so, any proper transfer function is admissible

Approximate PD Using Dynamic Compensation

Reminder: we can approximate the D-controller $K_{\rm D}s$ by

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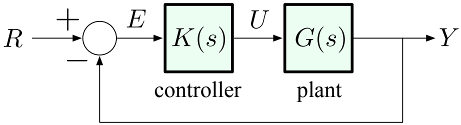

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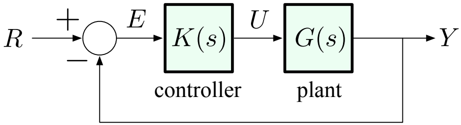

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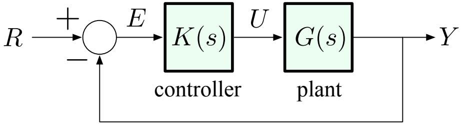

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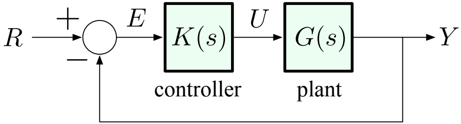

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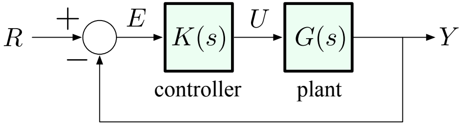

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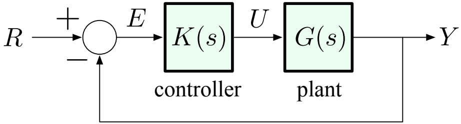

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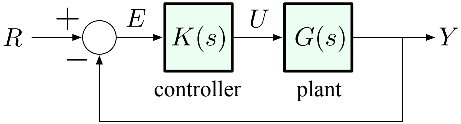

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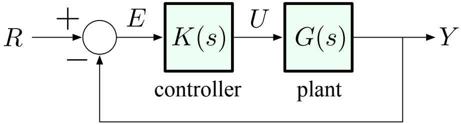

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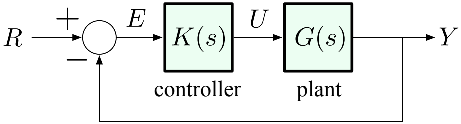

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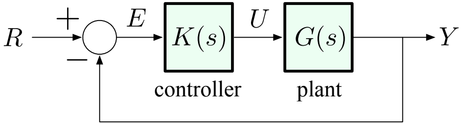
- the parameter $K = K_{\mathrm{P}} + pK_{\mathrm{D}}$ is a combination of P-gain, D-gain, and p

Approximate PD Using Dynamic Compensation



Closed-loop poles: $1 + \left(K_\mathrm{P} + K_\mathrm{D} \frac{ps}{s+p} \right) G(s) = 0$

Transform into Evans' canonical form:

$$
\begin{aligned}
K_\mathrm{P} + K_\mathrm{D} \frac{ps}{s+p} &= \frac{(K_\mathrm{P} + pK_\mathrm{D})s + pK_\mathrm{P}}{s+p} \\
&= (K_\mathrm{P} + pK_\mathrm{D}) \cdot \frac{s + \frac{pK_\mathrm{P}}{K_\mathrm{P} + pK_\mathrm{D}}}{s+p}
\end{aligned}
$$

Thus, we can write the controller as $K \cdot \frac{s+z}{s+p}$, where:

- the parameter $K = K_\mathrm{P} + pK_\mathrm{D}$ is a combination of P-gain, D-gain, and p
- the controller has an open-loop zero at $-z = -\frac{pK_\mathrm{P}}{K}$

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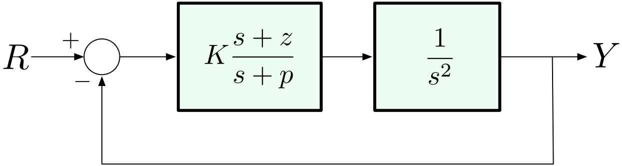
Double integrator:



Characteristic equation:

$$1 + K \cdot \frac{s+z}{s+p} \cdot \frac{1}{s^2} = 1 + KL(s) = 0$$

Note: $L(s)$ is *not* the open-loop transfer function; it comes from the forward gain shaped by the controller acting on the plant.