

Plan of the Lecture

- ▶ **Review:** basic properties and benefits of feedback control
- ▶ **Today's topic:** introduction to Proportional-Integral-Derivative (PID) control

Plan of the Lecture

- ▶ **Review:** basic properties and benefits of feedback control
- ▶ **Today's topic:** introduction to Proportional-Integral-Derivative (PID) control

Goal: study basic features and capabilities of PID control (industry standard since 1950's): arbitrary pole placement; reference tracking; disturbance rejection

Plan of the Lecture

- ▶ **Review:** basic properties and benefits of feedback control
- ▶ **Today's topic:** introduction to Proportional-Integral-Derivative (PID) control

Goal: study basic features and capabilities of PID control (industry standard since 1950's): arbitrary pole placement; reference tracking; disturbance rejection

Reading: FPE, Sections 4.1–4.3; lab manual

Recap: Benefits of Feedback Control

From last lecture: *feedback control*

- ▶ reduces steady-state error to disturbances
- ▶ reduces steady-state sensitivity to model uncertainty (parameter variations)
- ▶ improves time response

Recap: Benefits of Feedback Control

From last lecture: *feedback control*

- ▶ reduces steady-state error to disturbances
- ▶ reduces steady-state sensitivity to model uncertainty (parameter variations)
- ▶ improves time response

So far, we have only looked at *proportional feedback* (scalar gain) and 1st-order plants. Now we will add two more basic ingredients and examine their effect on higher-order systems.

Recap: Benefits of Feedback Control

From last lecture: *feedback control*

- ▶ reduces steady-state error to disturbances
- ▶ reduces steady-state sensitivity to model uncertainty (parameter variations)
- ▶ improves time response

So far, we have only looked at *proportional feedback* (scalar gain) and 1st-order plants. Now we will add two more basic ingredients and examine their effect on higher-order systems.

We will consider the following plant transfer function:

$$G(s) = \frac{1}{s^2 - 1}$$

Recap: Benefits of Feedback Control

From last lecture: *feedback control*

- ▶ reduces steady-state error to disturbances
- ▶ reduces steady-state sensitivity to model uncertainty (parameter variations)
- ▶ improves time response

So far, we have only looked at *proportional feedback* (scalar gain) and 1st-order plants. Now we will add two more basic ingredients and examine their effect on higher-order systems.

We will consider the following plant transfer function:

$$G(s) = \frac{1}{s^2 - 1}$$

- ▶ unstable: poles at $s = \pm 1$ (one pole in RHP)

Recap: Benefits of Feedback Control

From last lecture: *feedback control*

- ▶ reduces steady-state error to disturbances
- ▶ reduces steady-state sensitivity to model uncertainty (parameter variations)
- ▶ improves time response

So far, we have only looked at *proportional feedback* (scalar gain) and 1st-order plants. Now we will add two more basic ingredients and examine their effect on higher-order systems.

We will consider the following plant transfer function:

$$G(s) = \frac{1}{s^2 - 1}$$

- ▶ unstable: poles at $s = \pm 1$ (one pole in RHP)
- ▶ 2nd-order

Recap: Benefits of Feedback Control

From last lecture: *feedback control*

- ▶ reduces steady-state error to disturbances
- ▶ reduces steady-state sensitivity to model uncertainty (parameter variations)
- ▶ improves time response

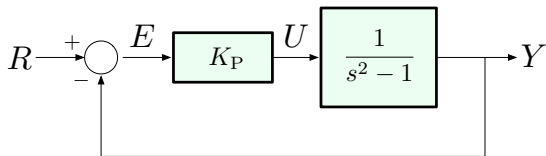
So far, we have only looked at *proportional feedback* (scalar gain) and 1st-order plants. Now we will add two more basic ingredients and examine their effect on higher-order systems.

We will consider the following plant transfer function:

$$G(s) = \frac{1}{s^2 - 1}$$

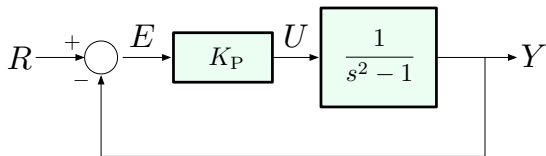
- ▶ unstable: poles at $s = \pm 1$ (one pole in RHP)
 - ▶ 2nd-order
- not as easy as DC motor, which was 1st-order and stable.

Proportional Feedback



K_P – “proportional gain” (P-gain) $U = K_P E$

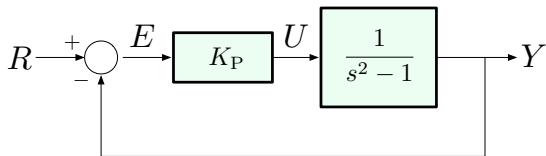
Proportional Feedback



K_P – “proportional gain” (P-gain) $U = K_P E$

Let's try to find a value of K_P that would stabilize the system:

Proportional Feedback

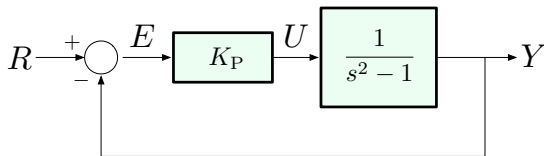


K_P – “proportional gain” (P-gain) $U = K_P E$

Let's try to find a value of K_P that would stabilize the system:

$$\frac{Y}{R}$$

Proportional Feedback

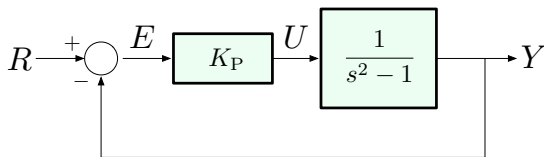


K_P – “proportional gain” (P-gain) $U = K_P E$

Let's try to find a value of K_P that would stabilize the system:

$$\frac{Y}{R} = \frac{\frac{K_P}{s^2 - 1}}{1 + \frac{K_P}{s^2 - 1}}$$

Proportional Feedback

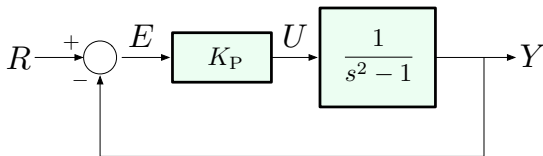


K_P – “proportional gain” (P-gain) $U = K_P E$

Let's try to find a value of K_P that would stabilize the system:

$$\frac{Y}{R} = \frac{\frac{K_P}{s^2 - 1}}{1 + \frac{K_P}{s^2 - 1}} = \frac{K_P}{s^2 - 1 + K_P}$$

Proportional Feedback



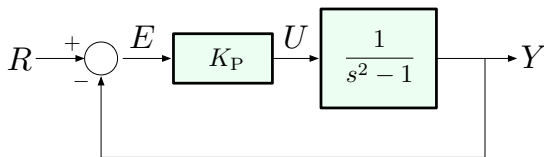
K_P – “proportional gain” (P-gain) $U = K_P E$

Let’s try to find a value of K_P that would stabilize the system:

$$\frac{Y}{R} = \frac{\frac{K_P}{s^2 - 1}}{1 + \frac{K_P}{s^2 - 1}} = \frac{K_P}{s^2 - 1 + K_P}$$

— the polynomial in the denominator has zero coefficient of s

Proportional Feedback



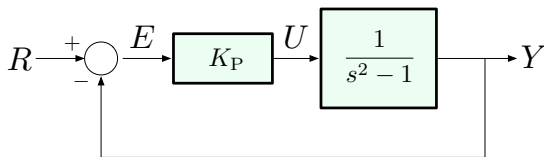
K_P – “proportional gain” (P-gain) $U = K_P E$

Let's try to find a value of K_P that would stabilize the system:

$$\frac{Y}{R} = \frac{\frac{K_P}{s^2 - 1}}{1 + \frac{K_P}{s^2 - 1}} = \frac{K_P}{s^2 - 1 + K_P}$$

— the polynomial in the denominator has zero coefficient of s
 \implies necessary condition for stability is not satisfied.

Proportional Feedback



K_P – “proportional gain” (P-gain) $U = K_P E$

Let's try to find a value of K_P that would stabilize the system:

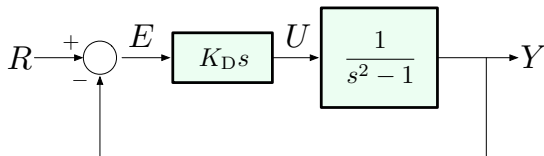
$$\frac{Y}{R} = \frac{\frac{K_P}{s^2 - 1}}{1 + \frac{K_P}{s^2 - 1}} = \frac{K_P}{s^2 - 1 + K_P}$$

— the polynomial in the denominator has zero coefficient of s
 \implies necessary condition for stability is not satisfied.

The feedback system is *not stable for any value of K_P !!*

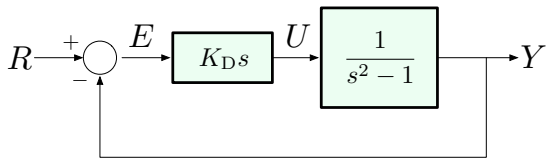
Derivative Feedback

Let's feed the *derivative of the error*, multiplied by some gain, back into the plant:



Derivative Feedback

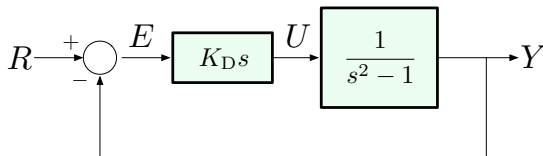
Let's feed the *derivative of the error*, multiplied by some gain, back into the plant:



Motivation: derivative = rate of change; faster change \implies more control needed.

Derivative Feedback

Let's feed the *derivative of the error*, multiplied by some gain, back into the plant:

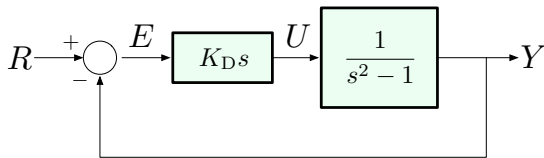


Motivation: derivative = rate of change; faster change \implies more control needed.

Caveat: multiplication by s is not a causal element

Derivative Feedback

Let's feed the *derivative of the error*, multiplied by some gain, back into the plant:

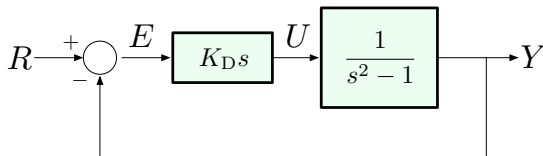


Motivation: derivative = rate of change; faster change \implies more control needed.

Caveat: multiplication by s is not a causal element (*why?*)

Derivative Feedback

Let's feed the *derivative of the error*, multiplied by some gain, back into the plant:



Motivation: derivative = rate of change; faster change \implies more control needed.

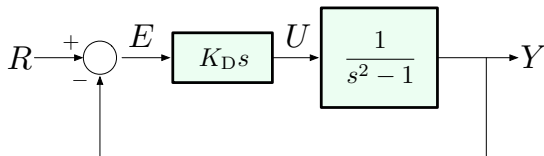
Caveat: multiplication by s is not a causal element (*why?*)

Derivative action and lack of causality: recall

$$\dot{e}(t) \approx \frac{e(t + \delta) - e(t)}{\delta} \quad (\text{for small } \delta)$$

Derivative Feedback

Let's feed the *derivative of the error*, multiplied by some gain, back into the plant:



Motivation: derivative = rate of change; faster change \implies more control needed.

Caveat: multiplication by s is not a causal element (**why?**)

Derivative action and lack of causality: recall

$$\dot{e}(t) \approx \frac{e(t + \delta) - e(t)}{\delta} \quad (\text{for small } \delta)$$

— if $\delta > 0$, $e(t + \delta)$ is in the future of $e(t)$!!

Disclaimer 1 about D-Feedback: Lack of Causality

Consider some state-space models:

Disclaimer 1 about D-Feedback: Lack of Causality

Consider some state-space models:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

Disclaimer 1 about D-Feedback: Lack of Causality

Consider some state-space models:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$$sX = AX + BU$$

$$Y = CX$$

Disclaimer 1 about D-Feedback: Lack of Causality

Consider some state-space models:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$$sX = AX + BU$$

$$Y = CX$$

$$(s - A)X = BU$$

Disclaimer 1 about D-Feedback: Lack of Causality

Consider some state-space models:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$$sX = AX + BU$$

$$Y = CX$$

$$(s - A)X = BU$$

$$\frac{Y}{U} = \frac{CB}{s - A}$$

Disclaimer 1 about D-Feedback: Lack of Causality

Consider some state-space models:

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

$$sX = AX + BU$$

$$Y = CX$$

$$(s - A)X = BU$$

$$\frac{Y}{U} = \frac{CB}{s - A} \equiv \frac{q(s)}{p(s)}$$

Disclaimer 1 about D-Feedback: Lack of Causality

Consider some state-space models:

$$\begin{array}{lll} \dot{x} = Ax + Bu & sX = AX + BU & (s - A)X = BU \\ y = Cx & Y = CX & \frac{Y}{U} = \frac{CB}{s - A} \equiv \frac{q(s)}{p(s)} \end{array}$$

$\deg(q) < \deg(p)$ — strictly proper transfer function

Disclaimer 1 about D-Feedback: Lack of Causality

Consider some state-space models:

$$\begin{array}{lll} \dot{x} = Ax + Bu & sX = AX + BU & (s - A)X = BU \\ y = Cx & Y = CX & \frac{Y}{U} = \frac{CB}{s - A} \equiv \frac{q(s)}{p(s)} \end{array}$$

$\deg(q) < \deg(p)$ — strictly proper transfer function

$$\begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx + Du \end{array}$$

Disclaimer 1 about D-Feedback: Lack of Causality

Consider some state-space models:

$$\begin{array}{lll} \dot{x} = Ax + Bu & sX = AX + BU & (s - A)X = BU \\ y = Cx & Y = CX & \frac{Y}{U} = \frac{CB}{s - A} \equiv \frac{q(s)}{p(s)} \end{array}$$

$\deg(q) < \deg(p)$ — strictly proper transfer function

$$\begin{array}{ll} \dot{x} = Ax + Bu & sX = AX + BU \\ y = Cx + Du & Y = CX + DU \end{array}$$

Disclaimer 1 about D-Feedback: Lack of Causality

Consider some state-space models:

$$\begin{array}{lll} \dot{x} = Ax + Bu & sX = AX + BU & (s - A)X = BU \\ y = Cx & Y = CX & \frac{Y}{U} = \frac{CB}{s - A} \equiv \frac{q(s)}{p(s)} \end{array}$$

$\deg(q) < \deg(p)$ — strictly proper transfer function

$$(s - A)X = BU$$

$$\begin{array}{ll} \dot{x} = Ax + Bu & sX = AX + BU \\ y = Cx + Du & Y = CX + DU \end{array}$$

Disclaimer 1 about D-Feedback: Lack of Causality

Consider some state-space models:

$$\begin{array}{lll} \dot{x} = Ax + Bu & sX = AX + BU & (s - A)X = BU \\ y = Cx & Y = CX & \frac{Y}{U} = \frac{CB}{s - A} \equiv \frac{q(s)}{p(s)} \end{array}$$

$\deg(q) < \deg(p)$ — strictly proper transfer function

$$\begin{array}{lll} \dot{x} = Ax + Bu & sX = AX + BU & (s - A)X = BU \\ y = Cx + Du & Y = CX + DU & Y = \frac{CB}{s - A}U + DU \end{array}$$

Disclaimer 1 about D-Feedback: Lack of Causality

Consider some state-space models:

$$\begin{array}{lll} \dot{x} = Ax + Bu & sX = AX + BU & (s - A)X = BU \\ y = Cx & Y = CX & \frac{Y}{U} = \frac{CB}{s - A} \equiv \frac{q(s)}{p(s)} \end{array}$$

$\deg(q) < \deg(p)$ — strictly proper transfer function

$$\begin{array}{lll} \dot{x} = Ax + Bu & sX = AX + BU & (s - A)X = BU \\ y = Cx + Du & Y = CX + DU & Y = \frac{CB}{s - A}U + DU \\ & & = \frac{CB + D(s - A)}{s - A}U \end{array}$$

Disclaimer 1 about D-Feedback: Lack of Causality

Consider some state-space models:

$$\begin{array}{lll} \dot{x} = Ax + Bu & sX = AX + BU & (s - A)X = BU \\ y = Cx & Y = CX & \frac{Y}{U} = \frac{CB}{s - A} \equiv \frac{q(s)}{p(s)} \end{array}$$

$\deg(q) < \deg(p)$ — strictly proper transfer function

$$\begin{array}{lll} \dot{x} = Ax + Bu & sX = AX + BU & (s - A)X = BU \\ y = Cx + Du & Y = CX + DU & Y = \frac{CB}{s - A}U + DU \\ & & = \frac{CB + D(s - A)}{s - A}U \equiv \frac{q(s)}{p(s)} \end{array}$$

Disclaimer 1 about D-Feedback: Lack of Causality

Consider some state-space models:

$$\begin{array}{lll} \dot{x} = Ax + Bu & sX = AX + BU & (s - A)X = BU \\ y = Cx & Y = CX & \frac{Y}{U} = \frac{CB}{s - A} \equiv \frac{q(s)}{p(s)} \end{array}$$

$\deg(q) < \deg(p)$ — strictly proper transfer function

$$\begin{array}{lll} \dot{x} = Ax + Bu & sX = AX + BU & (s - A)X = BU \\ y = Cx + Du & Y = CX + DU & Y = \frac{CB}{s - A}U + DU \\ & & = \frac{CB + D(s - A)}{s - A}U \equiv \frac{q(s)}{p(s)} \end{array}$$

$\deg(q) = \deg(p)$ — proper transfer function

Disclaimer 1 about D-Feedback: Lack of Causality

Consider some state-space models:

$$\begin{array}{lll} \dot{x} = Ax + Bu & sX = AX + BU & (s - A)X = BU \\ y = Cx & Y = CX & \frac{Y}{U} = \frac{CB}{s - A} \equiv \frac{q(s)}{p(s)} \end{array}$$

$\deg(q) < \deg(p)$ — strictly proper transfer function

$$\begin{array}{lll} \dot{x} = Ax + Bu & sX = AX + BU & (s - A)X = BU \\ y = Cx + Du & Y = CX + DU & Y = \frac{CB}{s - A}U + DU \\ & & = \frac{CB + D(s - A)}{s - A}U \equiv \frac{q(s)}{p(s)} \end{array}$$

$\deg(q) = \deg(p)$ — proper transfer function

Causal systems have proper transfer functions.

Lack of Causality

But if $u = K\dot{e}$, then $U = KsE$

Lack of Causality

$$\text{But if } u = K\dot{e}, \text{ then } U = KsE \implies \frac{U}{E} = Ks = \frac{q(s)}{p(s)}$$

Lack of Causality

But if $u = K\dot{e}$, then $U = KsE \implies \frac{U}{E} = Ks = \frac{q(s)}{p(s)}$

$$\deg(q) > \deg(p)$$

Lack of Causality

But if $u = K\dot{e}$, then $U = KsE \implies \frac{U}{E} = Ks = \frac{q(s)}{p(s)}$

$\deg(q) > \deg(p)$ — *improper system* (lack of causality)

Lack of Causality

But if $u = K\dot{e}$, then $U = KsE \implies \frac{U}{E} = Ks = \frac{q(s)}{p(s)}$

$\deg(q) > \deg(p)$ — *improper system* (lack of causality)

So, $E \mapsto K_D s E$ is not implementable directly, but we can implement an approximation, e.g.

$$\frac{K_D a s}{a + s} \longrightarrow K_D s \quad \text{as } a \rightarrow \infty$$

(this can be done using op-amps).

Lack of Causality

But if $u = K\dot{e}$, then $U = KsE \implies \frac{U}{E} = Ks = \frac{q(s)}{p(s)}$

$\deg(q) > \deg(p)$ — *improper system* (lack of causality)

So, $E \mapsto K_D s E$ is not implementable directly, but we can implement an approximation, e.g.

$$\frac{K_D a s}{a + s} \longrightarrow K_D s \quad \text{as } a \rightarrow \infty$$

(this can be done using op-amps).

Alternatively, we can approximate derivative action using finite differences:

$$\dot{e}(t) \approx \frac{e(t + \delta) - e(t)}{\delta},$$

Lack of Causality

But if $u = K\dot{e}$, then $U = KsE \implies \frac{U}{E} = Ks = \frac{q(s)}{p(s)}$

$\deg(q) > \deg(p)$ — *improper system* (lack of causality)

So, $E \mapsto K_D s E$ is not implementable directly, but we can implement an approximation, e.g.

$$\frac{K_D a s}{a + s} \longrightarrow K_D s \quad \text{as } a \rightarrow \infty$$

(this can be done using op-amps).

Alternatively, we can approximate derivative action using finite differences:

$$\dot{e}(t) \approx \frac{e(t + \delta) - e(t)}{\delta},$$

but then we must tolerate delays — must wait until time $t + \delta$ to issue a control signal meant for time t .

Disclaimer 2 about D-Feedback: Noise Amplification

Differentiators amplify noise (noise \rightarrow rapid changes in the reference).

Disclaimer 2 about D-Feedback: Noise Amplification

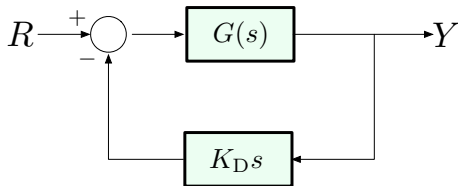
Differentiators amplify noise (noise \rightarrow rapid changes in the reference).

In the lab, D-feedback is implemented differently, in the feedback path. This way, we avoid differentiating the reference, which may be rapidly changing:

Disclaimer 2 about D-Feedback: Noise Amplification

Differentiators amplify noise (noise \rightarrow rapid changes in the reference).

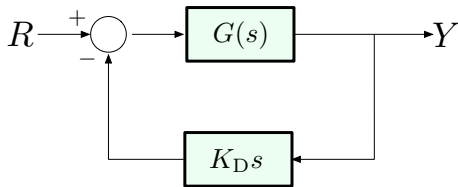
In the lab, D-feedback is implemented differently, in the feedback path. This way, we avoid differentiating the reference, which may be rapidly changing:



Disclaimer 2 about D-Feedback: Noise Amplification

Differentiators amplify noise (noise \rightarrow rapid changes in the reference).

In the lab, D-feedback is implemented differently, in the feedback path. This way, we avoid differentiating the reference, which may be rapidly changing:

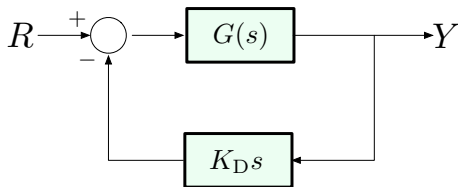


Before:
$$\frac{Y}{R} = \frac{K_D s G(s)}{1 + K_D s G(s)}$$

Disclaimer 2 about D-Feedback: Noise Amplification

Differentiators amplify noise (noise \rightarrow rapid changes in the reference).

In the lab, D-feedback is implemented differently, in the feedback path. This way, we avoid differentiating the reference, which may be rapidly changing:



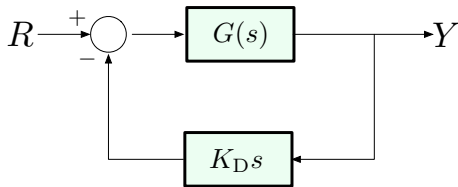
$$\text{Before: } \frac{Y}{R} = \frac{K_D s G(s)}{1 + K_D s G(s)}$$

$$\text{Now: } \frac{Y}{R} = \frac{G(s)}{1 + K_D s G(s)}$$

Disclaimer 2 about D-Feedback: Noise Amplification

Differentiators amplify noise (noise \rightarrow rapid changes in the reference).

In the lab, D-feedback is implemented differently, in the feedback path. This way, we avoid differentiating the reference, which may be rapidly changing:



$$\text{Before: } \frac{Y}{R} = \frac{K_D s G(s)}{1 + K_D s G(s)}$$

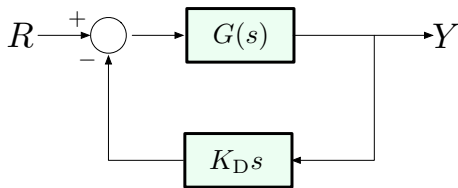
$$\text{Now: } \frac{Y}{R} = \frac{G(s)}{1 + K_D s G(s)}$$

$$\text{Poles: } 1 + K_D s G(s) = 0$$

Disclaimer 2 about D-Feedback: Noise Amplification

Differentiators amplify noise (noise \rightarrow rapid changes in the reference).

In the lab, D-feedback is implemented differently, in the feedback path. This way, we avoid differentiating the reference, which may be rapidly changing:



$$\text{Before: } \frac{Y}{R} = \frac{K_D s G(s)}{1 + K_D s G(s)}$$

$$\text{Now: } \frac{Y}{R} = \frac{G(s)}{1 + K_D s G(s)}$$

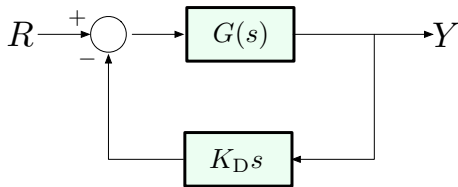
$$\text{Poles: } 1 + K_D s G(s) = 0$$

— same poles, but different zeros.

Disclaimer 2 about D-Feedback: Noise Amplification

Differentiators amplify noise (noise \rightarrow rapid changes in the reference).

In the lab, D-feedback is implemented differently, in the feedback path. This way, we avoid differentiating the reference, which may be rapidly changing:



$$\text{Before: } \frac{Y}{R} = \frac{K_D s G(s)}{1 + K_D s G(s)}$$

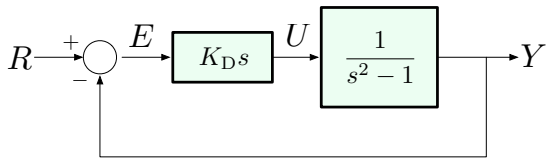
$$\text{Now: } \frac{Y}{R} = \frac{G(s)}{1 + K_D s G(s)}$$

$$\text{Poles: } 1 + K_D s G(s) = 0$$

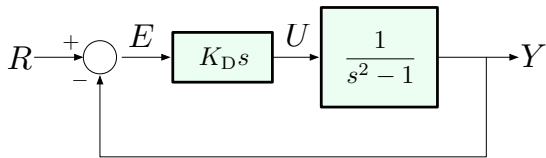
— same poles, but different zeros.

Now the reference signal is *smoothed out* by the plant $G(s)$ before entering the differentiator, which minimizes distortion due to noise.

Back to Analysis: Derivative Feedback

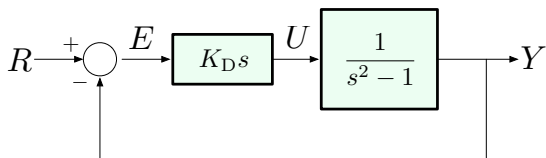


Back to Analysis: Derivative Feedback



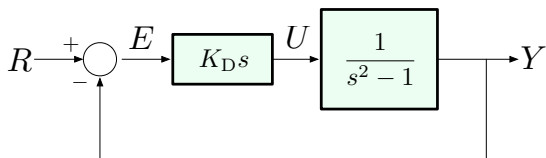
$$\frac{Y}{R}$$

Back to Analysis: Derivative Feedback



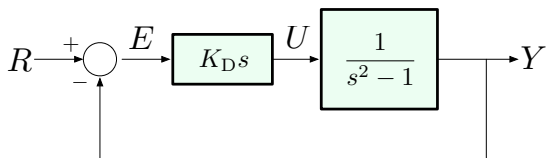
$$\frac{Y}{R} = \frac{\frac{K_D s}{s^2 - 1}}{1 + \frac{K_D s}{s^2 - 1}}$$

Back to Analysis: Derivative Feedback



$$\frac{Y}{R} = \frac{\frac{K_D s}{s^2 - 1}}{1 + \frac{K_D s}{s^2 - 1}} = \frac{K_D s}{s^2 + K_D s - 1}$$

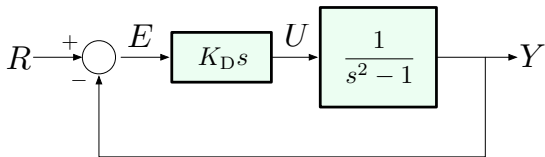
Back to Analysis: Derivative Feedback



$$\frac{Y}{R} = \frac{\frac{K_D s}{s^2 - 1}}{1 + \frac{K_D s}{s^2 - 1}} = \frac{K_D s}{s^2 + K_D s - 1}$$

— still not good: the denominator has a negative coefficient
 \implies not stable; also we have picked up a zero at the origin.

Back to Analysis: Derivative Feedback

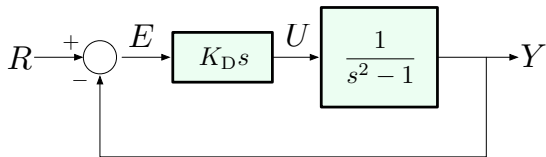


$$\frac{Y}{R} = \frac{\frac{K_D s}{s^2 - 1}}{1 + \frac{K_D s}{s^2 - 1}} = \frac{K_D s}{s^2 + K_D s - 1}$$

— still not good: the denominator has a negative coefficient
 \implies not stable; also we have picked up a zero at the origin.

But:

Back to Analysis: Derivative Feedback



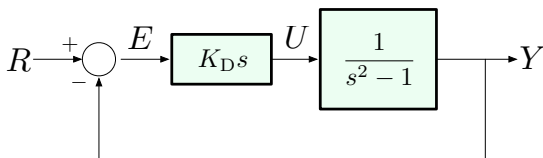
$$\frac{Y}{R} = \frac{\frac{K_D s}{s^2 - 1}}{1 + \frac{K_D s}{s^2 - 1}} = \frac{K_D s}{s^2 + K_D s - 1}$$

— still not good: the denominator has a negative coefficient
 \implies not stable; also we have picked up a zero at the origin.

But:

- ▶ P-control affected the coefficient of s^0 (constant term)

Back to Analysis: Derivative Feedback



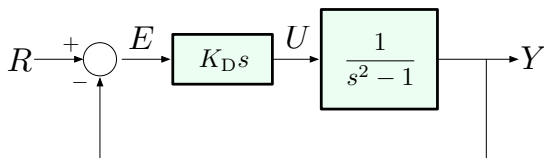
$$\frac{Y}{R} = \frac{\frac{K_D s}{s^2 - 1}}{1 + \frac{K_D s}{s^2 - 1}} = \frac{K_D s}{s^2 + K_D s - 1}$$

— still not good: the denominator has a negative coefficient
 \implies not stable; also we have picked up a zero at the origin.

But:

- ▶ P-control affected the coefficient of s^0 (constant term)
- ▶ D-control affected the coefficient of s

Back to Analysis: Derivative Feedback



$$\frac{Y}{R} = \frac{\frac{K_D s}{s^2 - 1}}{1 + \frac{K_D s}{s^2 - 1}} = \frac{K_D s}{s^2 + K_D s - 1}$$

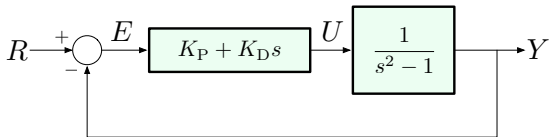
— still not good: the denominator has a negative coefficient
 \implies not stable; also we have picked up a zero at the origin.

But:

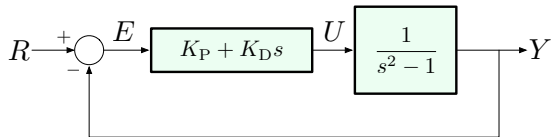
- ▶ P-control affected the coefficient of s^0 (constant term)
- ▶ D-control affected the coefficient of s

— let's combine them!!

Proportional-Derivative (PD) Control

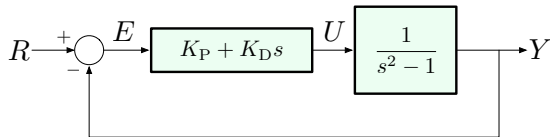


Proportional-Derivative (PD) Control



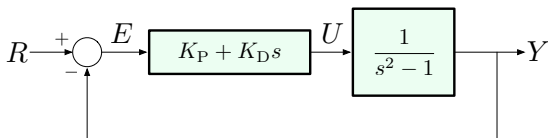
$$\frac{Y}{R}$$

Proportional-Derivative (PD) Control



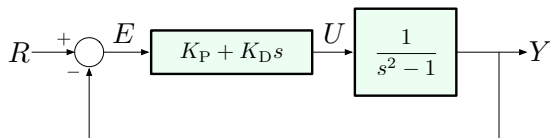
$$\frac{Y}{R} = \frac{\frac{K_P + K_D s}{s^2 - 1}}{1 + \frac{K_P + K_D s}{s^2 - 1}}$$

Proportional-Derivative (PD) Control



$$\frac{Y}{R} = \frac{\frac{K_P + K_D s}{s^2 - 1}}{1 + \frac{K_P + K_D s}{s^2 - 1}} = \frac{K_P + K_D s}{s^2 + K_D s + K_P - 1}$$

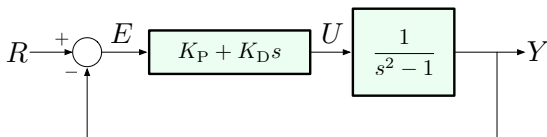
Proportional-Derivative (PD) Control



$$\frac{Y}{R} = \frac{\frac{K_P + K_D s}{s^2 - 1}}{1 + \frac{K_P + K_D s}{s^2 - 1}} = \frac{K_P + K_D s}{s^2 + K_D s + K_P - 1}$$

— now, if we set $K_D > 0$ and $K_P > 1$, then the transfer function will be stable.

Proportional-Derivative (PD) Control

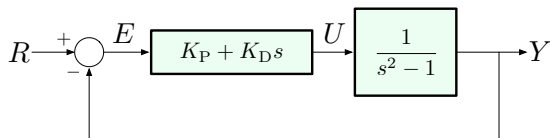


$$\frac{Y}{R} = \frac{\frac{K_P + K_D s}{s^2 - 1}}{1 + \frac{K_P + K_D s}{s^2 - 1}} = \frac{K_P + K_D s}{s^2 + K_D s + K_P - 1}$$

— now, if we set $K_D > 0$ and $K_P > 1$, then the transfer function will be stable.

Even more: by choosing K_P and K_D , we can *arbitrarily* assign coefficients of the denominator, and therefore the poles of the transfer function:

Proportional-Derivative (PD) Control



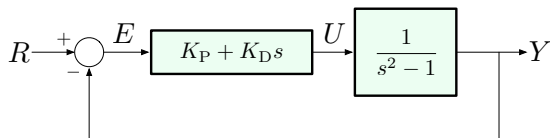
$$\frac{Y}{R} = \frac{\frac{K_P + K_D s}{s^2 - 1}}{1 + \frac{K_P + K_D s}{s^2 - 1}} = \frac{K_P + K_D s}{s^2 + K_D s + K_P - 1}$$

— now, if we set $K_D > 0$ and $K_P > 1$, then the transfer function will be stable.

Even more: by choosing K_P and K_D , we can *arbitrarily* assign coefficients of the denominator, and therefore the poles of the transfer function:

PD control gives us **arbitrary pole placement!!**

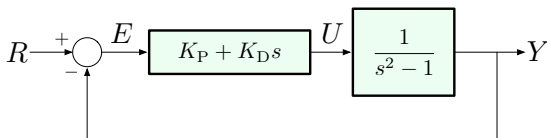
Proportional-Derivative (PD) Control



$$\frac{Y}{R} = \frac{K_P + K_D s}{s^2 + K_D s + K_P - 1}$$

By choosing K_P, K_D , we can achieve **arbitrary pole placement!!**

Proportional-Derivative (PD) Control

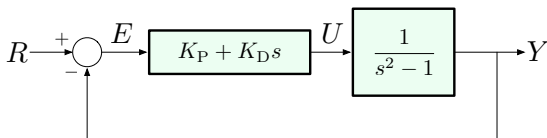


$$\frac{Y}{R} = \frac{K_P + K_D s}{s^2 + K_D s + K_P - 1}$$

By choosing K_P, K_D , we can achieve **arbitrary pole placement!!**

Also note that the addition of P-gain moves the zero:

Proportional-Derivative (PD) Control



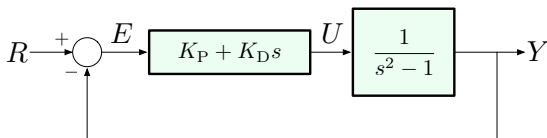
$$\frac{Y}{R} = \frac{K_P + K_D s}{s^2 + K_D s + K_P - 1}$$

By choosing K_P, K_D , we can achieve **arbitrary pole placement!!**

Also note that the addition of P-gain moves the zero:

$$K_D s + K_P = 0 \quad \text{LHP zero at } -\frac{K_P}{K_D}$$

Proportional-Derivative (PD) Control



$$\frac{Y}{R} = \frac{K_P + K_D s}{s^2 + K_D s + K_P - 1}$$

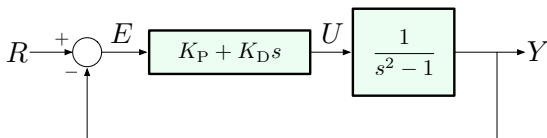
By choosing K_P, K_D , we can achieve **arbitrary pole placement!!**

Also note that the addition of P-gain moves the zero:

$$K_D s + K_P = 0 \quad \text{LHP zero at } -\frac{K_P}{K_D}$$

But what's missing?

Proportional-Derivative (PD) Control



$$\frac{Y}{R} = \frac{K_P + K_D s}{s^2 + K_D s + K_P - 1}$$

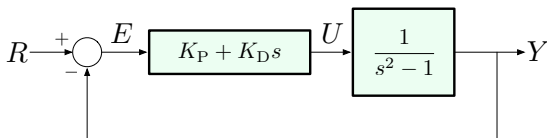
By choosing K_P, K_D , we can achieve **arbitrary pole placement!!**

Also note that the addition of P-gain moves the zero:

$$K_D s + K_P = 0 \quad \text{LHP zero at } -\frac{K_P}{K_D}$$

But what's missing? DC gain = $\left. \frac{Y}{R} \right|_{s=0} = \frac{K_P}{K_P - 1} \neq 1$

Proportional-Derivative (PD) Control



$$\frac{Y}{R} = \frac{K_P + K_D s}{s^2 + K_D s + K_P - 1}$$

By choosing K_P, K_D , we can achieve **arbitrary pole placement!!**

Also note that the addition of P-gain moves the zero:

$$K_D s + K_P = 0 \quad \text{LHP zero at } -\frac{K_P}{K_D}$$

But what's missing? DC gain = $\left. \frac{Y}{R} \right|_{s=0} = \frac{K_P}{K_P - 1} \neq 1$

— can't have perfect tracking of constant reference.

Proportional-Integral-Derivative (PID) Control

Let us try

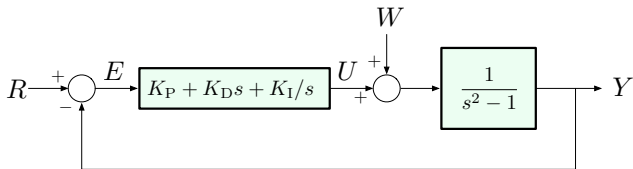
$$U = \left(K_P + K_D s + \frac{K_I}{s} \right) E \quad - \text{ the classic three-term controller}$$

Proportional-Integral-Derivative (PID) Control

Let us try

$$U = \left(K_P + K_D s + \frac{K_I}{s} \right) E \quad - \text{the classic three-term controller}$$

In fact, let's also throw in a constant disturbance:

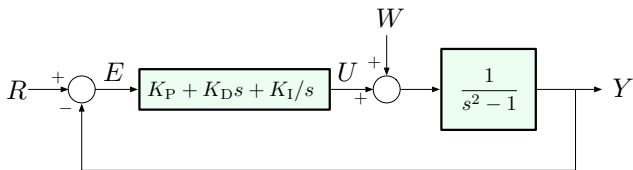


Proportional-Integral-Derivative (PID) Control

Let us try

$$U = \left(K_P + K_D s + \frac{K_I}{s} \right) E \quad - \text{the classic three-term controller}$$

In fact, let's also throw in a constant disturbance:

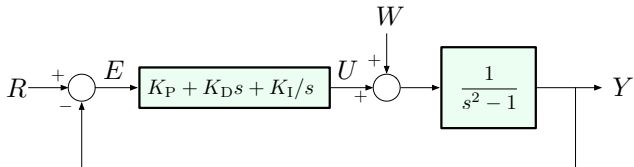


We will see that, with PID control, the goals of

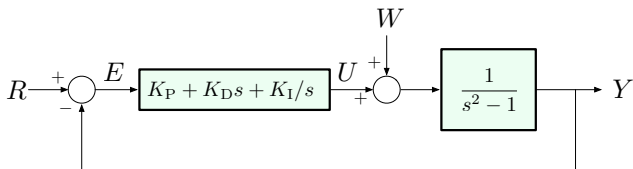
- ▶ tracking a constant reference r
- ▶ rejecting a constant disturbance w

can be accomplished in one shot.

Proportional-Integral-Derivative (PID) Control

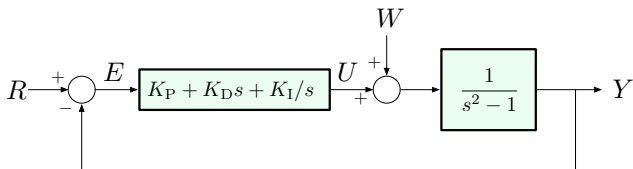


Proportional-Integral-Derivative (PID) Control



$$Y = \frac{1}{s^2 - 1}(U + W), \quad U = \left(K_P + K_D s + \frac{K_I}{s} \right) (R - Y)$$

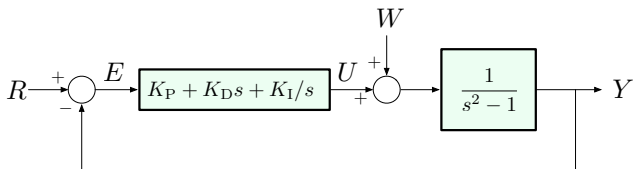
Proportional-Integral-Derivative (PID) Control



$$Y = \frac{1}{s^2 - 1}(U + W), \quad U = \left(K_P + K_D s + \frac{K_I}{s} \right) (R - Y)$$

$$\text{so } Y = \frac{K_P + K_D s + \frac{K_I}{s}}{s^2 - 1} (R - Y) + \frac{1}{s^2 - 1} W$$

Proportional-Integral-Derivative (PID) Control

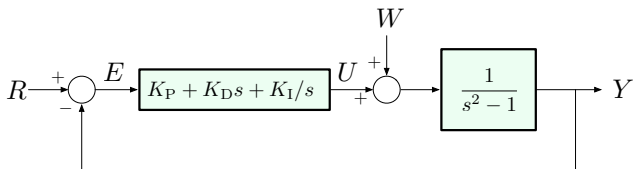


$$Y = \frac{1}{s^2 - 1}(U + W), \quad U = \left(K_P + K_D s + \frac{K_I}{s} \right) (R - Y)$$

$$\text{so } Y = \frac{K_P + K_D s + \frac{K_I}{s}}{s^2 - 1} (R - Y) + \frac{1}{s^2 - 1} W$$

Simplify:

Proportional-Integral-Derivative (PID) Control



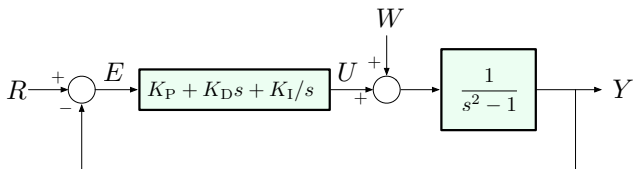
$$Y = \frac{1}{s^2 - 1}(U + W), \quad U = \left(K_P + K_D s + \frac{K_I}{s} \right) (R - Y)$$

$$\text{so } Y = \frac{K_P + K_D s + \frac{K_I}{s}}{s^2 - 1} (R - Y) + \frac{1}{s^2 - 1} W$$

Simplify:

$$(s^2 - 1)Y = \left(K_P + K_D s + \frac{K_I}{s} \right) (R - Y) + W$$

Proportional-Integral-Derivative (PID) Control



$$Y = \frac{1}{s^2 - 1}(U + W), \quad U = \left(K_P + K_D s + \frac{K_I}{s} \right) (R - Y)$$

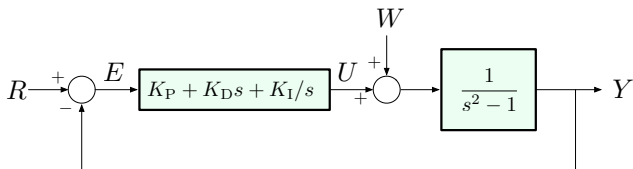
$$\text{so } Y = \frac{K_P + K_D s + \frac{K_I}{s}}{s^2 - 1} (R - Y) + \frac{1}{s^2 - 1} W$$

Simplify:

$$(s^2 - 1)Y = \left(K_P + K_D s + \frac{K_I}{s} \right) (R - Y) + W$$

$$\left(s^2 - 1 + K_P + K_D s + \frac{K_I}{s} \right) Y = \left(K_P + K_D s + \frac{K_I}{s} \right) R + W$$

Proportional-Integral-Derivative (PID) Control



$$Y = \frac{1}{s^2 - 1}(U + W), \quad U = \left(K_P + K_D s + \frac{K_I}{s} \right) (R - Y)$$

$$\text{so } Y = \frac{K_P + K_D s + \frac{K_I}{s}}{s^2 - 1} (R - Y) + \frac{1}{s^2 - 1} W$$

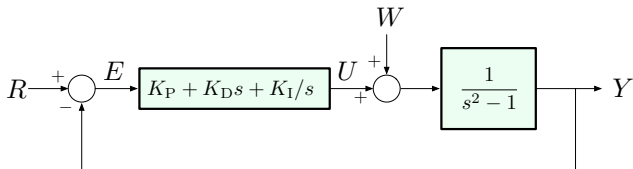
Simplify:

$$(s^2 - 1)Y = \left(K_P + K_D s + \frac{K_I}{s} \right) (R - Y) + W$$

$$\left(s^2 - 1 + K_P + K_D s + \frac{K_I}{s} \right) Y = \left(K_P + K_D s + \frac{K_I}{s} \right) R + W$$

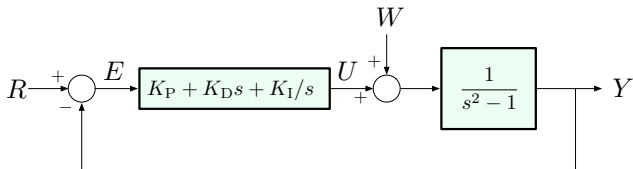
$$(s^3 - s + K_P s + K_D s^2 + K_I)Y = (K_P s + K_D s^2 + K_I)R + W s$$

Proportional-Integral-Derivative (PID) Control



$$(s^3 - s + K_P s + K_D s^2 + K_I)Y = (K_P s + K_D s^2 + K_I)R + W s$$

Proportional-Integral-Derivative (PID) Control

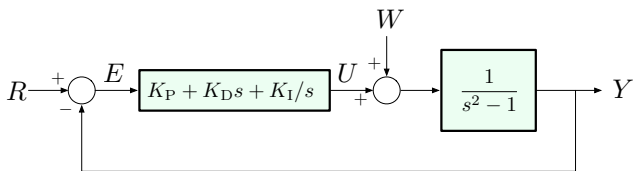


$$(s^3 - s + K_P s + K_D s^2 + K_I)Y = (K_P s + K_D s^2 + K_I)R + W s$$

Therefore,

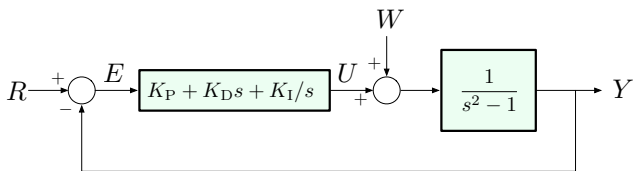
$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Proportional-Integral-Derivative (PID) Control



$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

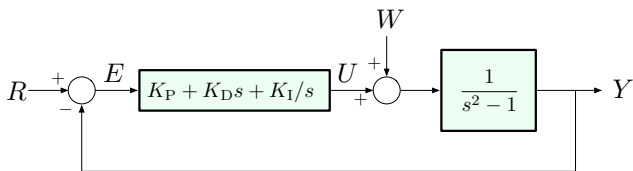
Proportional-Integral-Derivative (PID) Control



$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Stability:

Proportional-Integral-Derivative (PID) Control

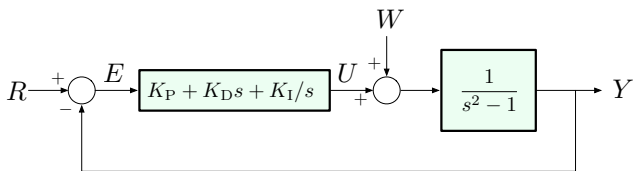


$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Stability:

- ▶ need $K_D > 0$, $K_P > 1$, $K_I > 0$ (necessary condition) and $K_D(K_P - 1) > K_I$ (Routh-Hurwitz for 3rd-order)

Proportional-Integral-Derivative (PID) Control

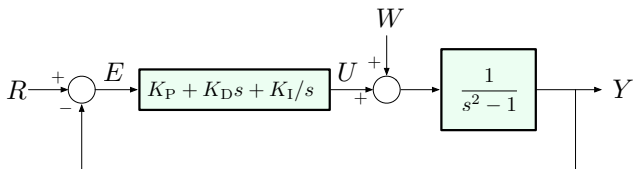


$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Stability:

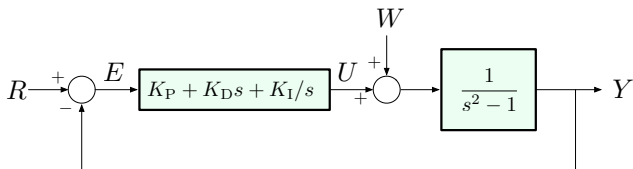
- ▶ need $K_D > 0$, $K_P > 1$, $K_I > 0$ (necessary condition) and $K_D(K_P - 1) > K_I$ (Routh-Hurwitz for 3rd-order)
- ▶ can still assign coefficients arbitrarily by choosing K_P, K_I, K_D

Proportional-Integral-Derivative (PID) Control



$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

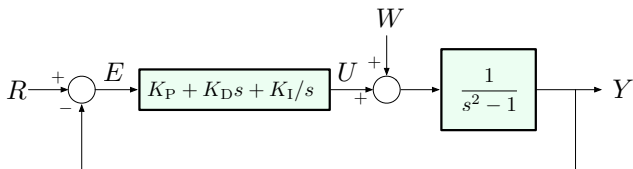
Proportional-Integral-Derivative (PID) Control



$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Reference tracking:

Proportional-Integral-Derivative (PID) Control

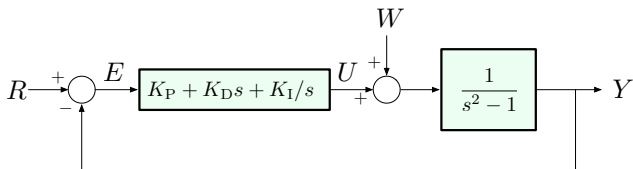


$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Reference tracking:

DC gain($R \rightarrow Y$)

Proportional-Integral-Derivative (PID) Control

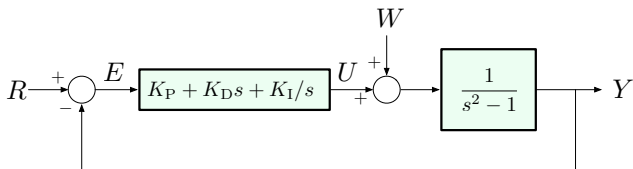


$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Reference tracking:

$$\text{DC gain}(R \rightarrow Y) = \left. \frac{K_D s^2 + K_P s + K_I}{s^3 + (K_P - 1)s + K_D s^2 + K_I} \right|_{s=0}$$

Proportional-Integral-Derivative (PID) Control

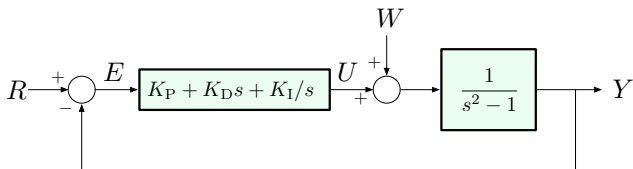


$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Reference tracking:

$$\text{DC gain}(R \rightarrow Y) = \left. \frac{K_D s^2 + K_P s + K_I}{s^3 + (K_P - 1)s + K_D s^2 + K_I} \right|_{s=0} = 1$$

Proportional-Integral-Derivative (PID) Control



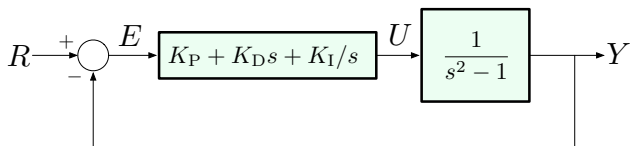
$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Reference tracking:

$$\text{DC gain}(R \rightarrow Y) = \frac{K_D s^2 + K_P s + K_I}{s^3 + (K_P - 1)s + K_D s^2 + K_I} \Bigg|_{s=0} = 1$$

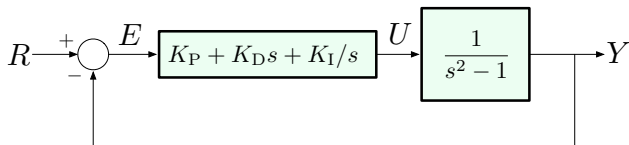
— so, with the addition of I-feedback, we remove earlier limitation and achieve *perfect tracking*!

Proportional-Integral-Derivative (PID) Control



$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

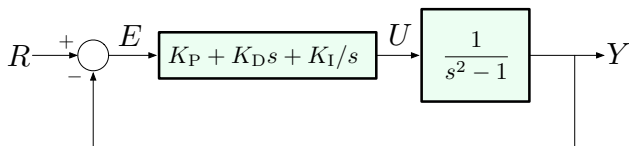
Proportional-Integral-Derivative (PID) Control



$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Disturbance rejection:

Proportional-Integral-Derivative (PID) Control

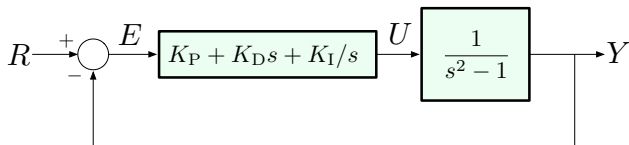


$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Disturbance rejection:

DC gain($W \rightarrow Y$)

Proportional-Integral-Derivative (PID) Control

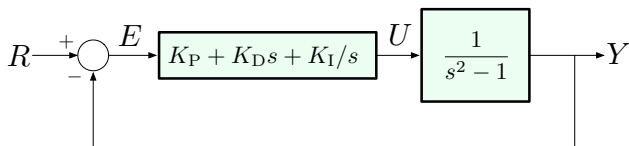


$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Disturbance rejection:

$$\text{DC gain}(W \rightarrow Y) = \left. \frac{s}{s^3 + (K_P - 1)s + K_D s^2 + K_I} \right|_{s=0}$$

Proportional-Integral-Derivative (PID) Control

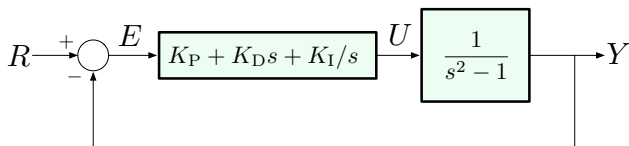


$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Disturbance rejection:

$$\text{DC gain}(W \rightarrow Y) = \left. \frac{s}{s^3 + (K_P - 1)s + K_D s^2 + K_I} \right|_{s=0} = 0$$

Proportional-Integral-Derivative (PID) Control



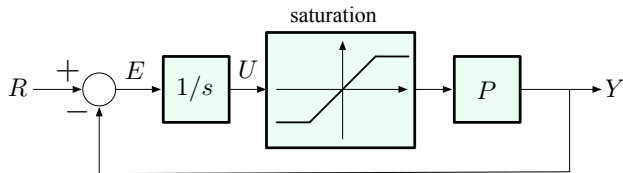
$$Y = \frac{K_D s^2 + K_P s + K_I}{s^3 + K_D s^2 + (K_P - 1)s + K_I} R + \frac{s}{s^3 + K_D s^2 + (K_P - 1)s + K_I} W$$

Disturbance rejection:

$$\text{DC gain}(W \rightarrow Y) = \left. \frac{s}{s^3 + (K_P - 1)s + K_D s^2 + K_I} \right|_{s=0} = 0$$

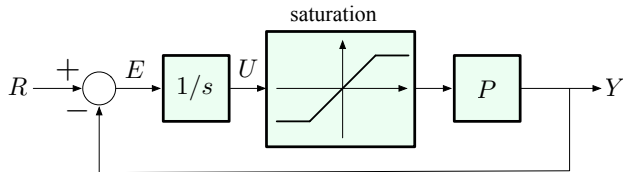
— so, integral gain also gives *complete attenuation* of *constant* disturbances!!

Wind-Up Phenomenon



When the actuator saturates, the error continues to be integrated, resulting in large overshoot.

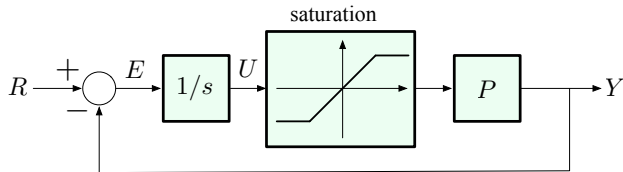
Wind-Up Phenomenon



When the actuator saturates, the error continues to be integrated, resulting in large overshoot.

We say that the integrator “winds up:” the error may be small, but its integrated past history builds up.

Wind-Up Phenomenon



When the actuator saturates, the error continues to be integrated, resulting in large overshoot.

We say that the integrator “winds up:” the error may be small, but its integrated past history builds up.

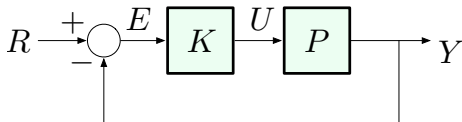
There are various *anti-windup* schemes to deal with this practically important issue. (Essentially, we attempt to detect the onset of saturation and turn the integrator off.)

System Type

The fact that $1/s$ leads to perfect tracking of constant references and perfect rejection of constant disturbances is a special case of a more general analysis.

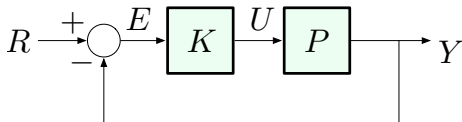
System Type

The fact that $1/s$ leads to perfect tracking of constant references and perfect rejection of constant disturbances is a special case of a more general analysis.



System Type

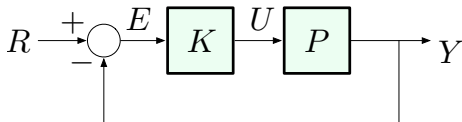
The fact that $1/s$ leads to perfect tracking of constant references and perfect rejection of constant disturbances is a special case of a more general analysis.



Consider the reference $r(t) = \frac{t^k}{k!}1(t) \longleftrightarrow R(s) = \frac{1}{s^{k+1}}$

System Type

The fact that $1/s$ leads to perfect tracking of constant references and perfect rejection of constant disturbances is a special case of a more general analysis.

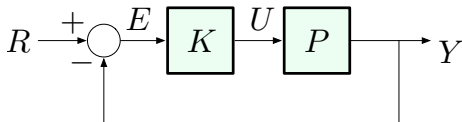


Consider the reference $r(t) = \frac{t^k}{k!}1(t) \longleftrightarrow R(s) = \frac{1}{s^{k+1}}$

Error signal: $E = \frac{1}{1 + KP}R = \frac{1}{1 + KP} \frac{1}{s^{k+1}}$

System Type

The fact that $1/s$ leads to perfect tracking of constant references and perfect rejection of constant disturbances is a special case of a more general analysis.



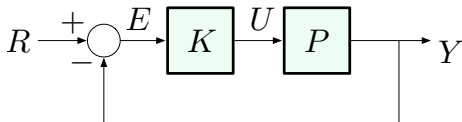
Consider the reference $r(t) = \frac{t^k}{k!} 1(t) \longleftrightarrow R(s) = \frac{1}{s^{k+1}}$

Error signal: $E = \frac{1}{1 + KP} R = \frac{1}{1 + KP} \frac{1}{s^{k+1}}$

FVT gives (assuming stability):

System Type

The fact that $1/s$ leads to perfect tracking of constant references and perfect rejection of constant disturbances is a special case of a more general analysis.



Consider the reference $r(t) = \frac{t^k}{k!} 1(t) \longleftrightarrow R(s) = \frac{1}{s^{k+1}}$

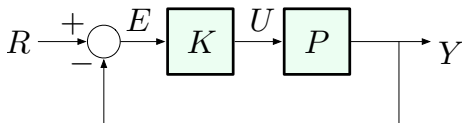
Error signal: $E = \frac{1}{1 + KP} R = \frac{1}{1 + KP} \frac{1}{s^{k+1}}$

FVT gives (assuming stability):

$$e(\infty)$$

System Type

The fact that $1/s$ leads to perfect tracking of constant references and perfect rejection of constant disturbances is a special case of a more general analysis.



Consider the reference $r(t) = \frac{t^k}{k!}1(t) \longleftrightarrow R(s) = \frac{1}{s^{k+1}}$

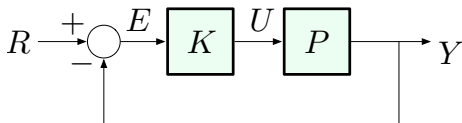
Error signal: $E = \frac{1}{1 + KP}R = \frac{1}{1 + KP} \frac{1}{s^{k+1}}$

FVT gives (assuming stability):

$$e(\infty) = sE(s) \Big|_{s=0}$$

System Type

The fact that $1/s$ leads to perfect tracking of constant references and perfect rejection of constant disturbances is a special case of a more general analysis.



Consider the reference $r(t) = \frac{t^k}{k!} 1(t) \longleftrightarrow R(s) = \frac{1}{s^{k+1}}$

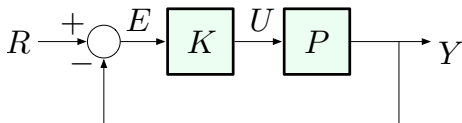
Error signal: $E = \frac{1}{1 + KP} R = \frac{1}{1 + KP} \frac{1}{s^{k+1}}$

FVT gives (assuming stability):

$$e(\infty) = sE(s) \Big|_{s=0} = \frac{1}{1 + KP} \frac{1}{s^k} \Big|_{s=0}$$

System Type

The fact that $1/s$ leads to perfect tracking of constant references and perfect rejection of constant disturbances is a special case of a more general analysis.



Consider the reference $r(t) = \frac{t^k}{k!}1(t) \longleftrightarrow R(s) = \frac{1}{s^{k+1}}$

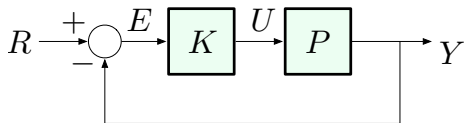
Error signal: $E = \frac{1}{1 + KP}R = \frac{1}{1 + KP} \frac{1}{s^{k+1}}$

FVT gives (assuming stability):

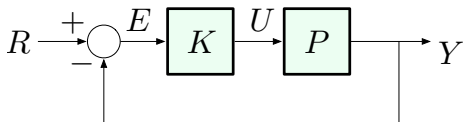
$$e(\infty) = sE(s) \Big|_{s=0} = \frac{1}{1 + KP} \frac{1}{s^k} \Big|_{s=0}$$

— let's see how the forward gain affects tracking performance.

System Type

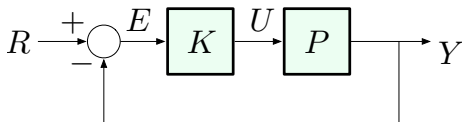


System Type



System type: the number n of poles the forward-loop transfer function KP has at the origin. It is the degree of the lowest-degree polynomial that cannot be tracked *in feedback* with zero steady-state error.

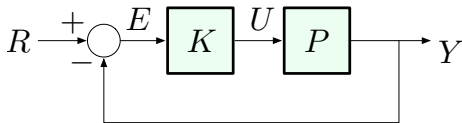
System Type



System type: the number n of poles the forward-loop transfer function KP has at the origin. It is the degree of the lowest-degree polynomial that cannot be tracked *in feedback* with zero steady-state error.

Note: the system type is calculated from the *forward-loop* transfer function, although the conclusions we will draw are about the *closed-loop system*.

System Type

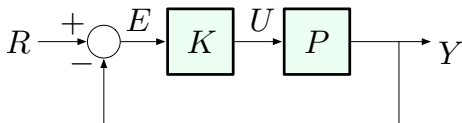


$$R(s) = \frac{1}{s^{k+1}} \implies E = \frac{1}{1 + KP} R = \frac{1}{1 + KP} \frac{1}{s^{k+1}}$$

$$e(\infty) = sE(s) \Big|_{s=0} = \frac{1}{1 + KP} \frac{1}{s^k} \Big|_{s=0}$$

— let's see how forward gain KP affects tracking performance.

System Type



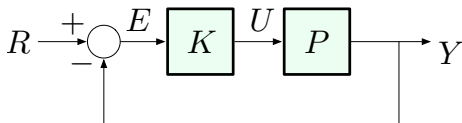
$$R(s) = \frac{1}{s^{k+1}} \implies E = \frac{1}{1 + KP} R = \frac{1}{1 + KP} \frac{1}{s^{k+1}}$$

$$e(\infty) = sE(s) \Big|_{s=0} = \frac{1}{1 + KP} \frac{1}{s^k} \Big|_{s=0}$$

— let's see how forward gain KP affects tracking performance.

Let's suppose that KP has n th-order pole at $s = 0$:

System Type



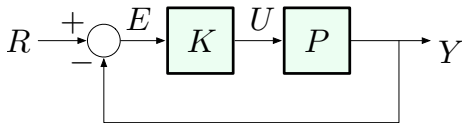
$$R(s) = \frac{1}{s^{k+1}} \implies E = \frac{1}{1 + KP} R = \frac{1}{1 + KP} \frac{1}{s^{k+1}}$$

$$e(\infty) = sE(s) \Big|_{s=0} = \frac{1}{1 + KP} \frac{1}{s^k} \Big|_{s=0}$$

— let's see how forward gain KP affects tracking performance.

Let's suppose that KP has n th-order pole at $s = 0$: $KP = \frac{K_0}{s^n}$

System Type



$$R(s) = \frac{1}{s^{k+1}} \implies E = \frac{1}{1 + KP} R = \frac{1}{1 + KP} \frac{1}{s^{k+1}}$$

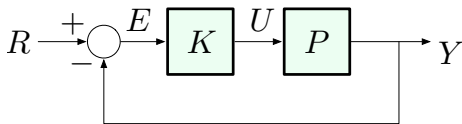
$$e(\infty) = sE(s) \Big|_{s=0} = \frac{1}{1 + KP} \frac{1}{s^k} \Big|_{s=0}$$

— let's see how forward gain KP affects tracking performance.

Let's suppose that KP has n th-order pole at $s = 0$: $KP = \frac{K_0}{s^n}$

$$sE(s)$$

System Type



$$R(s) = \frac{1}{s^{k+1}} \implies E = \frac{1}{1 + KP} R = \frac{1}{1 + KP} \frac{1}{s^{k+1}}$$

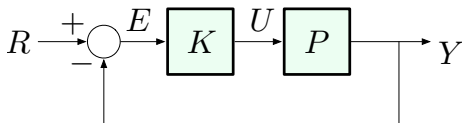
$$e(\infty) = sE(s) \Big|_{s=0} = \frac{1}{1 + KP} \frac{1}{s^k} \Big|_{s=0}$$

— let's see how forward gain KP affects tracking performance.

Let's suppose that KP has n th-order pole at $s = 0$: $KP = \frac{K_0}{s^n}$

$$sE(s) = \frac{1}{\left(1 + \frac{K_0}{s^n}\right) s^k}$$

System Type



$$R(s) = \frac{1}{s^{k+1}} \implies E = \frac{1}{1 + KP} R = \frac{1}{1 + KP} \frac{1}{s^{k+1}}$$

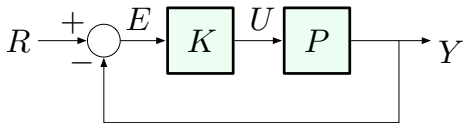
$$e(\infty) = sE(s) \Big|_{s=0} = \frac{1}{1 + KP} \frac{1}{s^k} \Big|_{s=0}$$

— let's see how forward gain KP affects tracking performance.

Let's suppose that KP has n th-order pole at $s = 0$: $KP = \frac{K_0}{s^n}$

$$sE(s) = \frac{1}{\left(1 + \frac{K_0}{s^n}\right) s^k} = \frac{s^{n-k}}{s^n + K_0}$$

System Type



$$R(s) = \frac{1}{s^{k+1}} \implies E = \frac{1}{1 + KP} R = \frac{1}{1 + KP} \frac{1}{s^{k+1}}$$

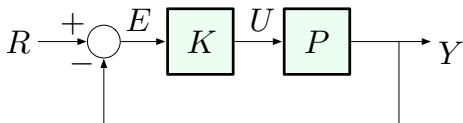
$$e(\infty) = sE(s) \Big|_{s=0} = \frac{1}{1 + KP} \frac{1}{s^k} \Big|_{s=0}$$

— let's see how forward gain KP affects tracking performance.

Let's suppose that KP has n th-order pole at $s = 0$: $KP = \frac{K_0}{s^n}$

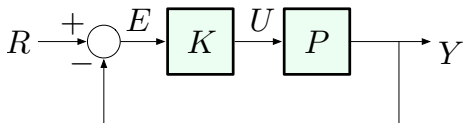
$$sE(s) = \frac{1}{\left(1 + \frac{K_0}{s^n}\right) s^k} = \frac{s^{n-k}}{s^n + K_0} \quad \text{— what about } sE(s) \Big|_{s=0} ?$$

System Type



Let's suppose that KP has n th-order pole at $s = 0$: $KP = \frac{K_0}{s^n}$

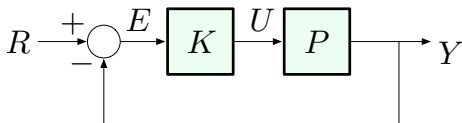
System Type



Let's suppose that KP has n th-order pole at $s = 0$: $KP = \frac{K_0}{s^n}$

$$sE(s) = \frac{1}{\left(1 + \frac{K_0}{s^n}\right) s^k} = \frac{s^{n-k}}{s^n + K_0} \quad \text{— what about } sE(s)\Big|_{s=0} ?$$

System Type

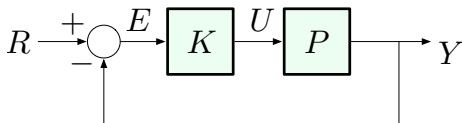


Let's suppose that KP has n th-order pole at $s = 0$: $KP = \frac{K_0}{s^n}$

$$sE(s) = \frac{1}{\left(1 + \frac{K_0}{s^n}\right) s^k} = \frac{s^{n-k}}{s^n + K_0} \quad \text{--- what about } sE(s) \Big|_{s=0} ?$$

Recall: reference $r(t)$ is a polynomial of degree k

System Type



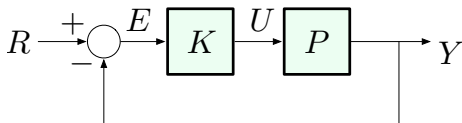
Let's suppose that KP has n th-order pole at $s = 0$: $KP = \frac{K_0}{s^n}$

$$sE(s) = \frac{1}{\left(1 + \frac{K_0}{s^n}\right) s^k} = \frac{s^{n-k}}{s^n + K_0} \quad \text{--- what about } sE(s)\Big|_{s=0}?$$

Recall: reference $r(t)$ is a polynomial of degree k

Three cases to consider —

System Type



Let's suppose that KP has n th-order pole at $s = 0$: $KP = \frac{K_0}{s^n}$

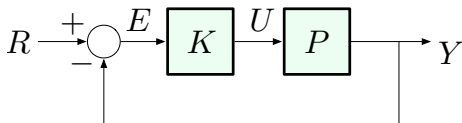
$$sE(s) = \frac{1}{\left(1 + \frac{K_0}{s^n}\right) s^k} = \frac{s^{n-k}}{s^n + K_0} \quad \text{— what about } sE(s) \Big|_{s=0} ?$$

Recall: reference $r(t)$ is a polynomial of degree k

Three cases to consider —

- ▶ $n > k$:

System Type



Let's suppose that KP has n th-order pole at $s = 0$: $KP = \frac{K_0}{s^n}$

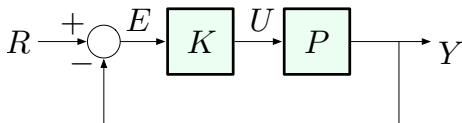
$$sE(s) = \frac{1}{\left(1 + \frac{K_0}{s^n}\right) s^k} = \frac{s^{n-k}}{s^n + K_0} \quad \text{— what about } sE(s) \Big|_{s=0} ?$$

Recall: reference $r(t)$ is a polynomial of degree k

Three cases to consider —

- ▶ $n > k$: $e(\infty) = 0$

System Type



Let's suppose that KP has n th-order pole at $s = 0$: $KP = \frac{K_0}{s^n}$

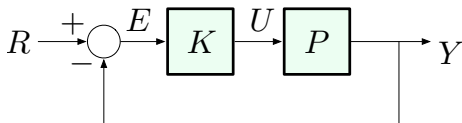
$$sE(s) = \frac{1}{\left(1 + \frac{K_0}{s^n}\right) s^k} = \frac{s^{n-k}}{s^n + K_0} \quad \text{— what about } sE(s) \Big|_{s=0} ?$$

Recall: reference $r(t)$ is a polynomial of degree k

Three cases to consider —

- ▶ $n > k$: $e(\infty) = 0$ perfect tracking

System Type



Let's suppose that KP has n th-order pole at $s = 0$: $KP = \frac{K_0}{s^n}$

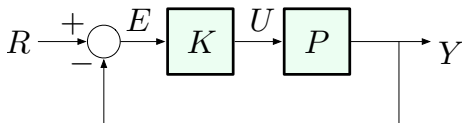
$$sE(s) = \frac{1}{\left(1 + \frac{K_0}{s^n}\right) s^k} = \frac{s^{n-k}}{s^n + K_0} \quad \text{— what about } sE(s) \Big|_{s=0} ?$$

Recall: reference $r(t)$ is a polynomial of degree k

Three cases to consider —

- ▶ $n > k$: $e(\infty) = 0$ perfect tracking
- ▶ $n = k$:

System Type



Let's suppose that KP has n th-order pole at $s = 0$: $KP = \frac{K_0}{s^n}$

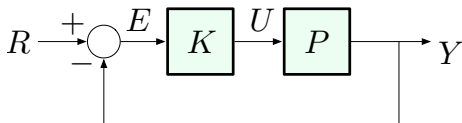
$$sE(s) = \frac{1}{\left(1 + \frac{K_0}{s^n}\right) s^k} = \frac{s^{n-k}}{s^n + K_0} \quad \text{— what about } sE(s) \Big|_{s=0} ?$$

Recall: reference $r(t)$ is a polynomial of degree k

Three cases to consider —

- ▶ $n > k$: $e(\infty) = 0$ perfect tracking
- ▶ $n = k$: $e(\infty) = \text{const} \neq 0$

System Type



Let's suppose that KP has n th-order pole at $s = 0$: $KP = \frac{K_0}{s^n}$

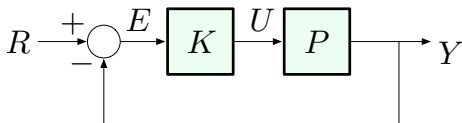
$$sE(s) = \frac{1}{\left(1 + \frac{K_0}{s^n}\right) s^k} = \frac{s^{n-k}}{s^n + K_0} \quad \text{— what about } sE(s) \Big|_{s=0} ?$$

Recall: reference $r(t)$ is a polynomial of degree k

Three cases to consider —

- ▶ $n > k$: $e(\infty) = 0$ perfect tracking
- ▶ $n = k$: $e(\infty) = \text{const} \neq 0$ imperfect tracking

System Type



Let's suppose that KP has n th-order pole at $s = 0$: $KP = \frac{K_0}{s^n}$

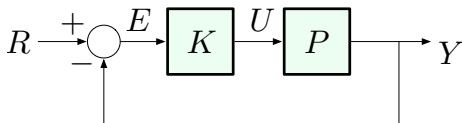
$$sE(s) = \frac{1}{\left(1 + \frac{K_0}{s^n}\right) s^k} = \frac{s^{n-k}}{s^n + K_0} \quad \text{--- what about } sE(s) \Big|_{s=0} ?$$

Recall: reference $r(t)$ is a polynomial of degree k

Three cases to consider —

- ▶ $n > k$: $e(\infty) = 0$ perfect tracking
- ▶ $n = k$: $e(\infty) = \text{const} \neq 0$ imperfect tracking
- ▶ $n < k$:

System Type



Let's suppose that KP has n th-order pole at $s = 0$: $KP = \frac{K_0}{s^n}$

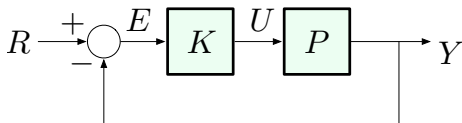
$$sE(s) = \frac{1}{\left(1 + \frac{K_0}{s^n}\right) s^k} = \frac{s^{n-k}}{s^n + K_0} \quad \text{— what about } sE(s) \Big|_{s=0} ?$$

Recall: reference $r(t)$ is a polynomial of degree k

Three cases to consider —

- ▶ $n > k$: $e(\infty) = 0$ perfect tracking
- ▶ $n = k$: $e(\infty) = \text{const} \neq 0$ imperfect tracking
- ▶ $n < k$: $e(\infty) = \infty$

System Type



Let's suppose that KP has n th-order pole at $s = 0$: $KP = \frac{K_0}{s^n}$

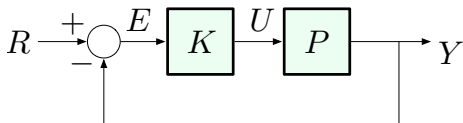
$$sE(s) = \frac{1}{\left(1 + \frac{K_0}{s^n}\right) s^k} = \frac{s^{n-k}}{s^n + K_0} \quad \text{— what about } sE(s) \Big|_{s=0} ?$$

Recall: reference $r(t)$ is a polynomial of degree k

Three cases to consider —

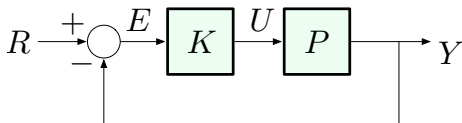
- ▶ $n > k$: $e(\infty) = 0$ perfect tracking
- ▶ $n = k$: $e(\infty) = \text{const} \neq 0$ imperfect tracking
- ▶ $n < k$: $e(\infty) = \infty$ no tracking

System Type: Examples



System type is the degree of the lowest-degree polynomial that cannot be tracked *in feedback* with zero steady-state error.

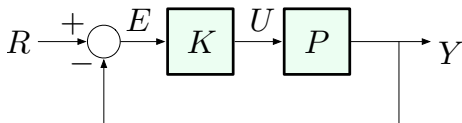
System Type: Examples



System type is the degree of the lowest-degree polynomial that cannot be tracked *in feedback* with zero steady-state error.

- ▶ **Type 0:** no pole at the origin. This is what we had without the I-gain: nonzero SS error to constant references.

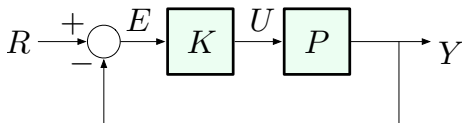
System Type: Examples



System type is the degree of the lowest-degree polynomial that cannot be tracked *in feedback* with zero steady-state error.

- ▶ **Type 0:** no pole at the origin. This is what we had without the I-gain: nonzero SS error to constant references.
- ▶ **Type 1:** a single pole at the origin. This is what we get with I-gain: can track (respectively, reject) constant references (respectively, disturbances) with zero error.

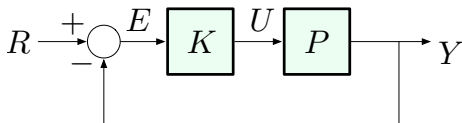
System Type: Examples



System type is the degree of the lowest-degree polynomial that cannot be tracked *in feedback* with zero steady-state error.

- ▶ **Type 0:** no pole at the origin. This is what we had without the I-gain: nonzero SS error to constant references.
- ▶ **Type 1:** a single pole at the origin. This is what we get with I-gain: can track (respectively, reject) constant references (respectively, disturbances) with zero error.
 - ▶ can check that we have a nonzero (but finite) error when tracking ramp references

System Type: Examples



System type is the degree of the lowest-degree polynomial that cannot be tracked *in feedback* with zero steady-state error.

- ▶ **Type 0:** no pole at the origin. This is what we had without the I-gain: nonzero SS error to constant references.
- ▶ **Type 1:** a single pole at the origin. This is what we get with I-gain: can track (respectively, reject) constant references (respectively, disturbances) with zero error.
 - ▶ can check that we have a nonzero (but finite) error when tracking ramp references
- ▶ **Type 2:** a double pole at the origin. Can track ramp references without error, but not t^2, t^3, \dots

PID Control: Summary & Further Comments

PID Control: Summary & Further Comments

P-gain simplest to implement, but not always sufficient for stabilization

PID Control: Summary & Further Comments

P-gain simplest to implement, but not always sufficient for stabilization

D-gain helps achieve stability, improves time response (more control over pole locations)

PID Control: Summary & Further Comments

P-gain simplest to implement, but not always sufficient for stabilization

D-gain helps achieve stability, improves time response (more control over pole locations)

- ▶ arbitrary pole placement only valid for 2nd-order response; in general, we still have control over two *dominant poles*

PID Control: Summary & Further Comments

P-gain simplest to implement, but not always sufficient for stabilization

D-gain helps achieve stability, improves time response (more control over pole locations)

- ▶ arbitrary pole placement only valid for 2nd-order response; in general, we still have control over two *dominant poles*
- ▶ cannot be implemented directly, so need approximate implementation; D-gain also amplifies noise

PID Control: Summary & Further Comments

P-gain simplest to implement, but not always sufficient for stabilization

D-gain helps achieve stability, improves time response (more control over pole locations)

- ▶ arbitrary pole placement only valid for 2nd-order response; in general, we still have control over two *dominant poles*
- ▶ cannot be implemented directly, so need approximate implementation; D-gain also amplifies noise

I-gain essential for perfect steady-state tracking of constant reference and rejection of constant disturbance

PID Control: Summary & Further Comments

P-gain simplest to implement, but not always sufficient for stabilization

D-gain helps achieve stability, improves time response (more control over pole locations)

- ▶ arbitrary pole placement only valid for 2nd-order response; in general, we still have control over two *dominant poles*
- ▶ cannot be implemented directly, so need approximate implementation; D-gain also amplifies noise

I-gain essential for perfect steady-state tracking of constant reference and rejection of constant disturbance

- ▶ but $1/s$ is not a stable element by itself, so one must be careful: it can destabilize the system if the feedback loop is broken (**integrator wind-up**)