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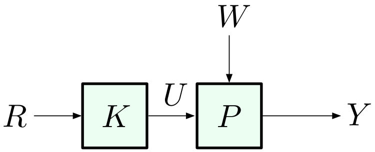
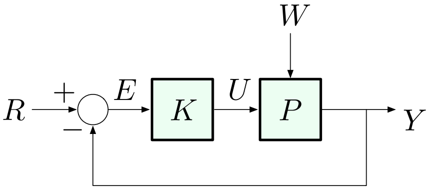
Reading: FPE, Section 4.1; lab manual

Two Basic Control Architectures

- Open-loop control

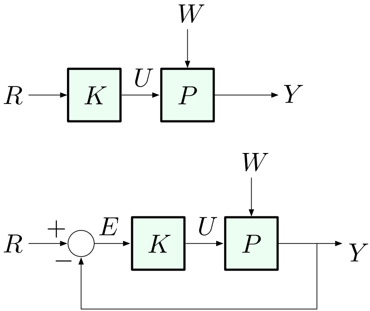


- Feedback (closed-loop) control



Here, W is a *disturbance*; K is *not necessarily* a static gain

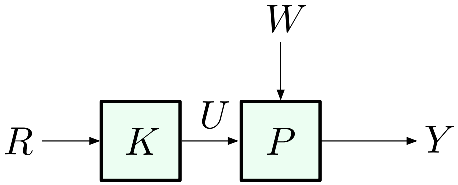
Basic Objectives of Control



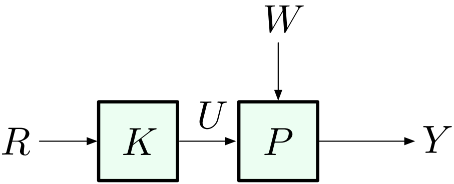
- track a given reference
- reject disturbances
- meet performance specs

Intuitively, the difference between the open-loop and the closed-loop architectures is clear (think cruise control ...)

Open-Loop Control

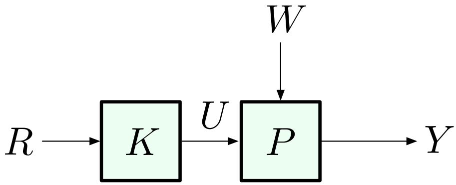


Open-Loop Control



- cheaper/easier to implement (no sensor required)

Open-Loop Control



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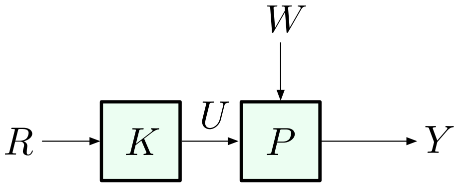

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  e.g., if both K and P are stable (all poles in OLHP),

$$\frac{Y}{R} = KP$$

  is also stable:

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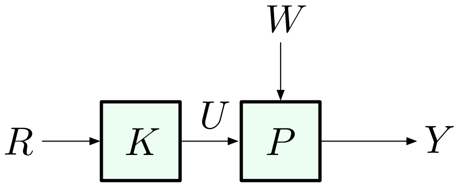

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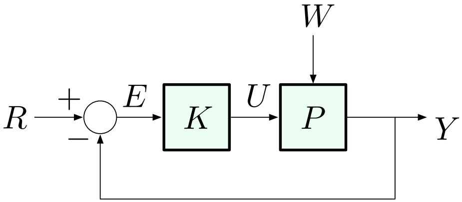
$$\{\text{poles of } KP\} = \{\text{poles of } K\} \cup \{\text{poles of } P\}$$

Feedback Control



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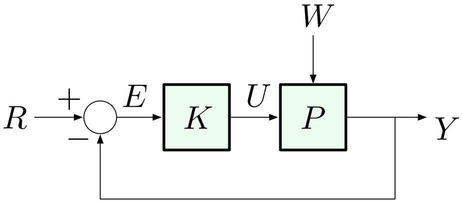

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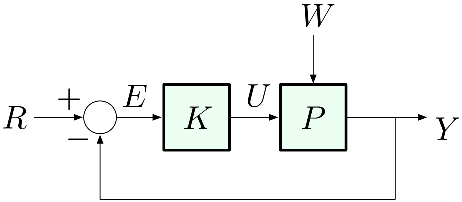

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has new poles, which may be unstable

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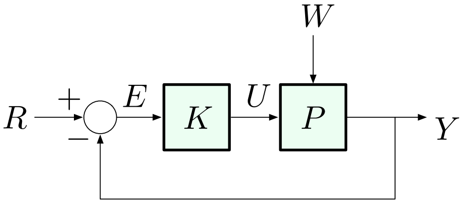

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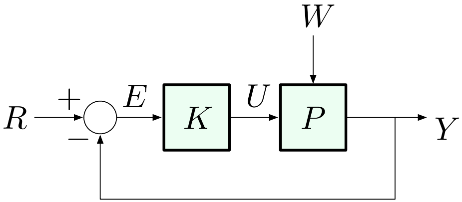
  has new poles, which may be unstable
- but: feedback control is the *only way* to stabilize an unstable plant (this was the Wright brothers' key insight)

Benefits of Feedback Control



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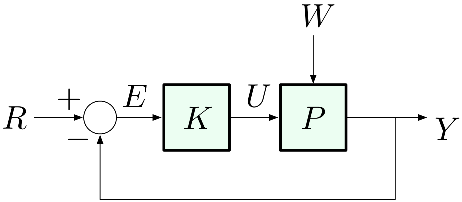
Benefits of Feedback Control



Feedback control:

- reduces steady-state error to disturbances

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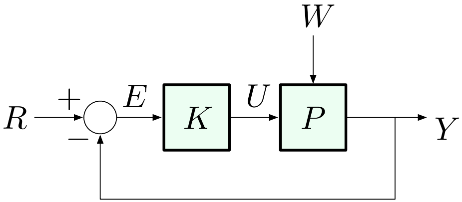

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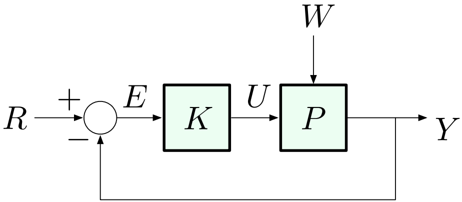

Feedback control:

- reduces steady-state error to disturbances
- reduces steady-state sensitivity to model uncertainty (parameter variations)
- improves time response

Case Study: DC Motor

Inputs:	v_{a} – input voltage
	τ_{e} – load/disturbance torque
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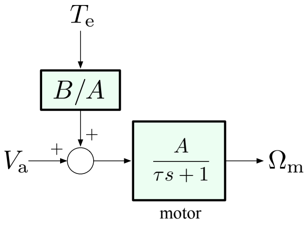

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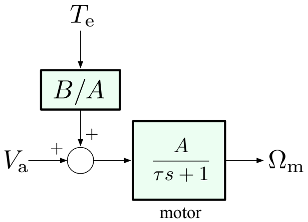
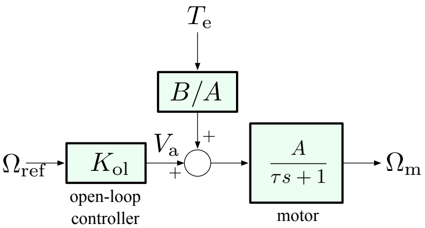


Objective: have Ω_{m} approach and track a given reference Ω_{ref} in spite of disturbance T_{e}.

Two Control Configurations

- Open-loop control



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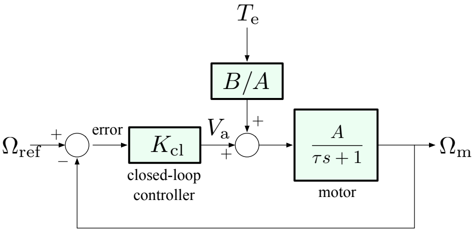

Disturbance Rejection

Goal: maintain $\omega_{\mathrm{m}} = \omega_{\mathrm{ref}}$ in steady state in the presence of *constant* disturbance.

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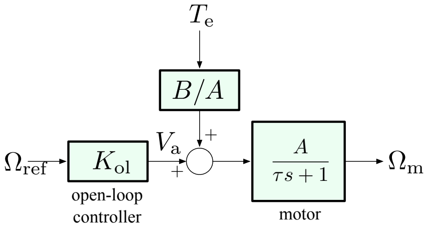

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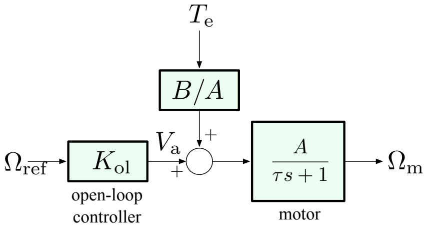

– the controller receives *no information* about the disturbance τ_e (the only input is ω_ref, no feedback signal from anywhere else)

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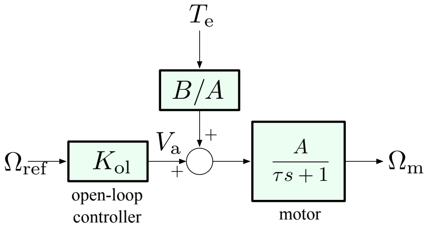
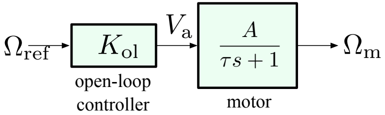


– the controller receives *no information* about the disturbance $\tau_{\rm e}$ (the only input is $\omega_{\rm ref}$, no feedback signal from anywhere else)

– so, let's attempt the following: design for *no disturbance* (i.e., $\tau_{\rm e} = 0$), then see how the system works in general

Disturbance Rejection: Open-Loop Control

First assume zero disturbance:



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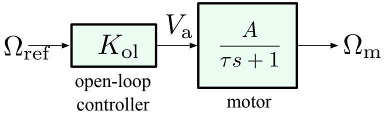

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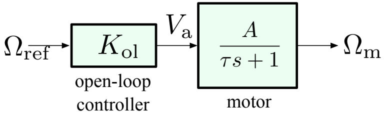

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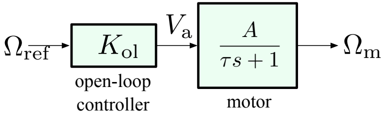

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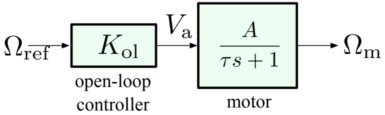

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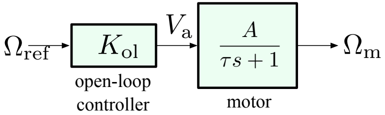

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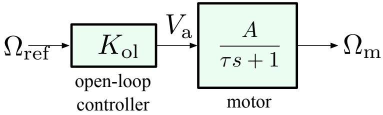

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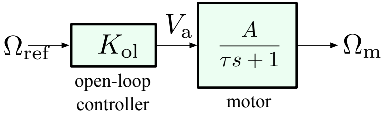

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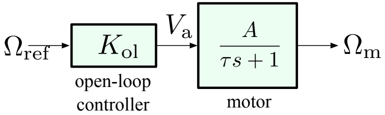

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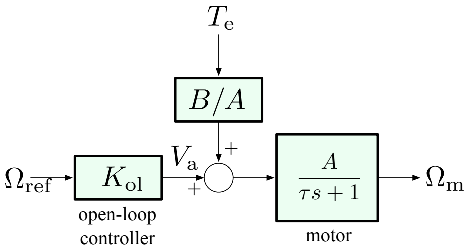
$$\Longrightarrow \omega_{\text{m}}(\infty) = \underbrace{\omega_{\text{ref}}}_{\text{step input}} + B \underbrace{\tau_{\text{e}}}_{\text{step input}}$$

Disturbance Rejection: Open-Loop Control

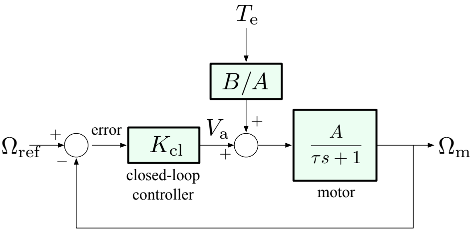
Steady-state motor speed for constant reference and constant disturbance:

$$\omega_{\mathrm{m}}(\infty) = \omega_{\mathrm{ref}} + B\tau_{\mathrm{e}}$$



Conclusion: in the absence of disturbances, reference tracking is good, but disturbance rejection is pretty poor. Steady-state error is determined by B, and we have no control over it (and, in fact, cannot change this through any choice of controller K_{ol}, no matter how clever)

Disturbance Rejection: Feedback Control



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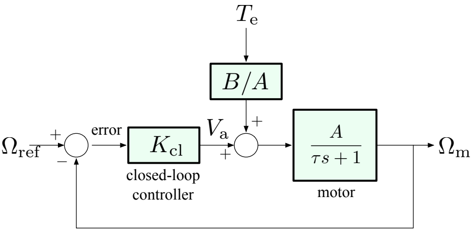

$$V_a = K_{cl} E$$

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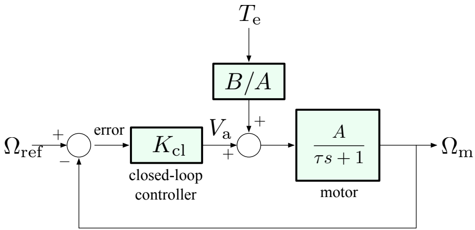

$$V_a = K_{cl} E = K_{cl} \left(\Omega_{ref} - \Omega_m \right)$$

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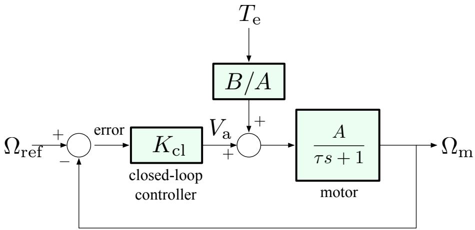

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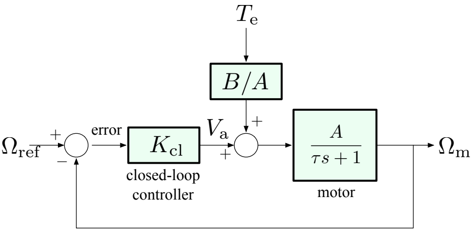

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Solve for Ω_m:

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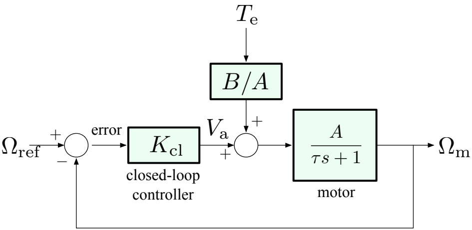

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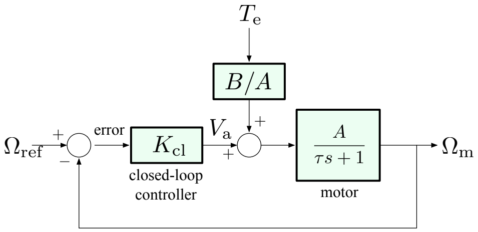

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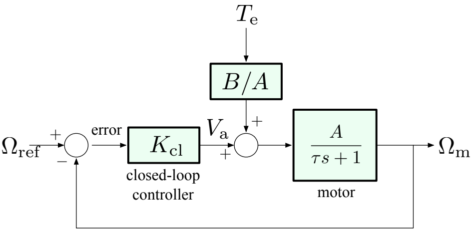

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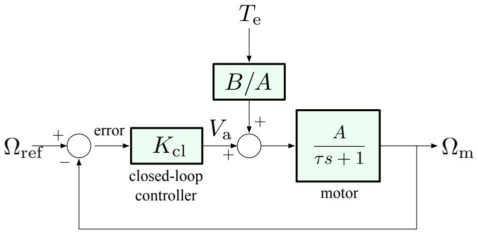
$$\Omega_m = \frac{AK_{cl}}{\tau s + 1 + AK_{cl}}\Omega_{ref} + \frac{B}{\tau s + 1 + AK_{cl}}T_e$$

Disturbance Rejection: Feedback Control



$$\Omega_m = \underbrace{\frac{AK_{cl}}{\tau s + 1 + AK_{cl}}}_{\text{DC gain}=\frac{AK_{cl}}{1+AK_{cl}}} \Omega_{ref} + \underbrace{\frac{B}{\tau s + 1 + AK_{cl}}}_{\text{DC gain}=\frac{B}{1+AK_{cl}}} T_e$$

(provided all transfer functions are strictly stable)

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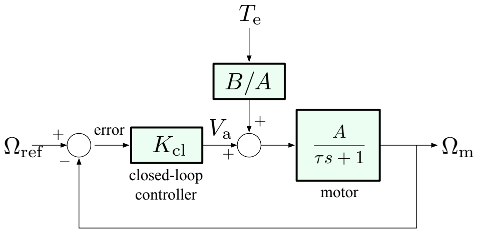

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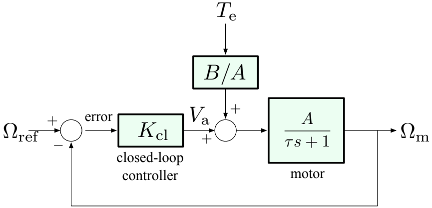
Assuming that the reference ω_{ref} and the disturbance τ_e are constant, we apply FVT:

$$\omega_m(\infty) = \frac{AK_{cl}}{1 + AK_{cl}}\omega_{ref} + \frac{B}{1 + AK_{cl}}\tau_e$$

Disturbance Rejection: Feedback Control

Steady-state speed for constant reference and disturbance:



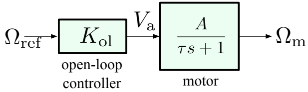
$$\omega_m(\infty) = \frac{AK_{cl}}{1+AK_{cl}}\omega_{ref} + \frac{B}{1+AK_{cl}}\tau_e$$

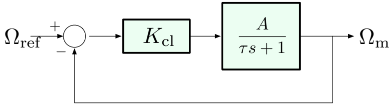
Conclusions:

- $\dfrac{AK_{cl}}{1+AK_{cl}} \neq 1$, but can be brought arbitrarily close to 1 when $K_{cl} \to \infty$. Thus, steady-state tracking is good with high gain, but never quite as good as in open-loop case.
- $\dfrac{B}{1+AK_{cl}}$ is small (arbitrarily close to 0) for large K_{cl}. Thus, *much* better disturbance rejection than with open-loop control.

Sensitivity to Parameter Variations

Consider again our DC motor model, with no disturbance:





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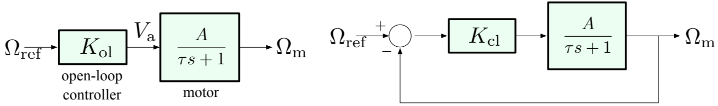

Bode's sensitivity concept: In the "nominal" situation, we have the motor with DC gain $= A$, and the overall transfer function, either open- or closed-loop, has some other DC gain (call it T).

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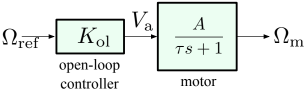

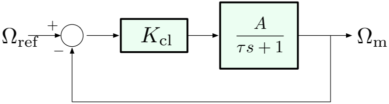

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Now suppose that, due to modeling error, changes in operating conditions, etc., the motor gain changes:

$$A \longrightarrow A + \underbrace{\delta A}_{\substack{\text{small} \\ \text{perturbation}}}$$

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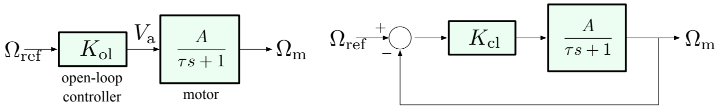

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This will cause a perturbation in the overall DC gain:

$$T \longrightarrow T + \delta T \qquad \text{(from calculus, to 1st order, } \delta T \approx \frac{\mathrm{d}T}{\mathrm{d}A}\delta A\text{)}$$

Sensitivity to Parameter Variations

$$A \longrightarrow A + \delta A \qquad \text{(small perturbation in system gain)}$$

$$T \longrightarrow T + \delta T \qquad \text{(resultant perturbation in overall DC gain)}$$

Hendrik Wade Bode
(1905–1982)

Bode's sensitivity:

$$\mathcal{S} \triangleq \frac{\delta T / T}{\delta A / A}$$

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$$\begin{aligned} \mathcal{S} &= \text{relative error} \\ &= \frac{\text{normalized (percentage) error in } T}{\text{normalized (percentage) error in } A} \end{aligned}$$

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Let's compute $\mathcal{S}$ for our DC motor control example, both open- and closed-loop.

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$$T_{\text{ol}} \longrightarrow K_{\text{ol}}(A + \delta A)$$

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For example, a 5% error in A will cause a 5% error in T_{ol}.

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Taylor expansion:

$$T(A + \delta A) = T(A) + \frac{\mathrm{d}T}{\mathrm{d}A}(A)\delta A + \text{higher-order terms}$$

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With high-gain feedback, we get smaller relative error due to parameter variations in the plant model.

Time Response

We still assume no disturbance: $\tau_e = 0$.

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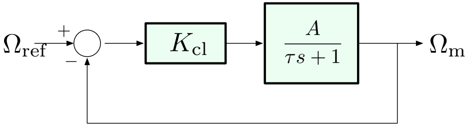
Here, τ is the *time constant*: the time it takes the system response to decay to $1/e$ of its starting value.

In the open-loop case, larger time constant means faster convergence to steady state. This is not affected by the choice of K_{cl} in any way!

Time Response

Closed-loop



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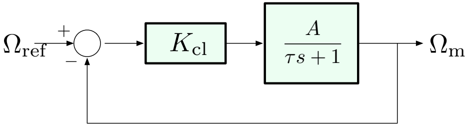

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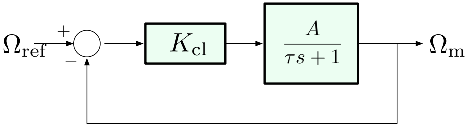

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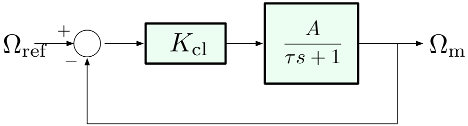

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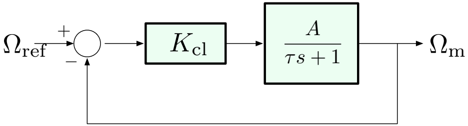

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— for large K_{cl}, we have a much smaller time constant, i.e., *faster convergence* to steady-state.

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Feedback control:

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Thus, we need a more sophisticated design than just static gain. We will take this up in the next lecture with *Proportional-Integral-Derivative* (PID) control.