NOTE: You don’t need to submit this problem set, it is just to help you prepare for the final exam. Solutions will be posted on the web.

Reading: FPE, Sections 7.6 and 7.10.2.

Problems:

1. (exam material) In class we derived the closed-loop system obtained with dynamic output feedback in \((x, \hat{x})\)-coordinates:
\[
\begin{pmatrix}
\dot{x} \\
\dot{\hat{x}}
\end{pmatrix} = \begin{pmatrix} A & -BK \\ LC & A - LC - BK \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}
\]
and later rewrote it in \((x, e)\)-coordinates. Rewrite the same system in \((\hat{x}, e)\)-coordinates.

2. (exam material) Consider the plant transfer function \(G(s) = \frac{1}{s(s + 1)}\).

a) Find any controllable and observable state-space realization of \(G(s)\).

b) Stabilize the state-space system from part a) by dynamic output feedback. Select arbitrary controller and observer poles such that the closed-loop system is stable and has reasonable damping (in your judgement).

c) Compute the transfer function of the controller you found in part b). Write it in the form \(kD(s)\), where \(k\) is a scalar gain (not to be confused with the state feedback gain matrix \(K\)) and \(D(s)\) is a ratio of monic polynomials (leading coefficients equal 1).

d) Draw the (positive) root locus for \(L(s) = D(s)G(s)\) and find on it the locations of the closed-loop poles you chose in part b).

e) Draw the Bode plot for \(kD(s)G(s)\) and compute the gain margin and phase margin.

f) Decide whether you’re happy with the closed-loop system. If not, go back and improve the design.

3. (not exam material) Consider the system
\[
\begin{align*}
\dot{x}_1 &= x_1 + x_2 \\
\dot{x}_2 &= -x_1 + x_2 + u \\
y &= 2x_1 + x_2
\end{align*}
\]
and suppose that the control objective is to minimize the performance index \(\int_0^\infty [\rho y^2(t) + u^2(t)] dt\), \(\rho > 0\).

a) Show graphically the locations of the optimal closed-loop poles as the parameter \(\rho\) varies (symmetric root locus).

b) See why in the limit as \(\rho \to 0\) (“expensive control” case), the optimal closed-loop poles become mirror images of the open-loop poles across the imaginary axis.

c) See why in the limit as \(\rho \to \infty\) (“cheap control” case), one optimal closed-loop pole cancels the open-loop zero and the other moves off to \(-\infty\).