Plan of the Lecture

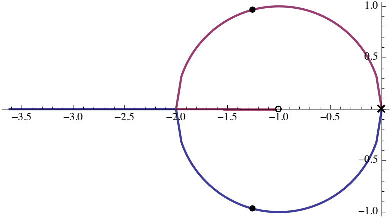
- Review: rules for sketching root loci; introduction to dynamic compensation
- Today's topic: lead and lag dynamic compensation

Goal: introduce the use of lead and lag dynamic compensators for approximate implementation of PD and PI control.

Reading: FPE, Chapter 5

From Last Time: Double Integrator with PD-Control

Characteristic equation: $$1 + K \cdot \frac{s+1}{s^2} = 0$$



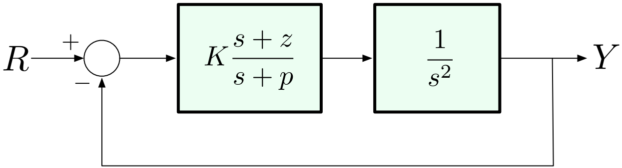
What can we conclude from this root locus about stabilization?

- all closed-loop poles are in LHP (we already knew this from Routh, but now can visualize)
- nice damping, so can meet reasonable specs

So, the effect of D-gain was to introduce an *open-loop zero* into LHP, and this zero "pulled" the root locus into LHP, thus stabilizing the system.

Dynamic Compensation

Objectives: stabilize the system and satisfy given time response specs using a *stable*, *causal* controller.



Characteristic equation:

$$1 + K \cdot \frac{s+z}{s+p} \cdot \frac{1}{s^2} = 1 + KL(s) = 0$$

Approximate PD Using Dynamic Compensation

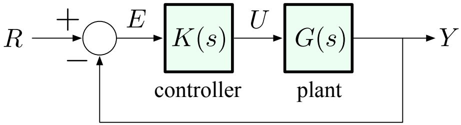
Reminder: we can approximate the D-controller $K_{\mathrm{D}}s$ by

$$K_{\mathrm{D}}\frac{ps}{s+p} \longrightarrow K_{\mathrm{D}}s \text{ as } p \to \infty$$

— here, $-p$ is the *pole* of the controller.

So, we replace the PD controller $K_{\mathrm{P}} + K_{\mathrm{D}}s$ by

$$K(s) = K_{\mathrm{P}} + K_{\mathrm{D}}\frac{ps}{s+p}$$



Closed-loop poles: $1 + \left(K_{\mathrm{P}} + K_{\mathrm{D}}\frac{ps}{s+p}\right)G(s) = 0$

Lead & Lag Compensators

Consider a general controller of the form

$$K\frac{s+z}{s+p} \qquad \text{— } K, z, p > 0 \text{ are design parameters}$$

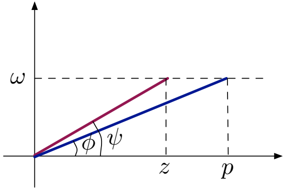
Depending on the relative values of z and p, we call it:

- a lead compensator when $z < p$
- a lag compensator when $z > p$

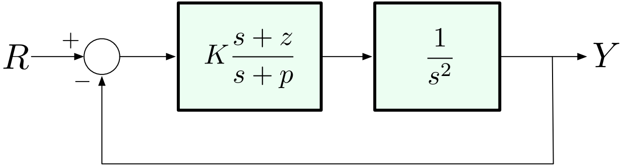
Why the name "lead/lag?" — think frequency response

$$\angle\frac{j\omega+z}{j\omega+p} = \angle(j\omega+z) - \angle(j\omega+p) = \psi - \phi$$

- if $z < p$, then $\psi - \phi > 0$ (phase lead)
- if $z > p$, then $\psi - \phi < 0$ (phase lag)



Back to Double Integrator



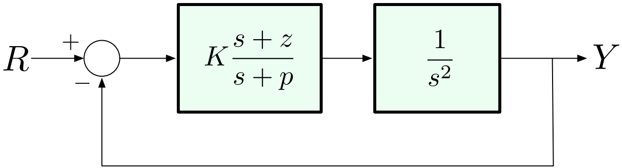
Controller transfer function is $K\dfrac{s+z}{s+p}$, where:

$$K = K_\mathrm{P} + pK_\mathrm{D}, \qquad z = \frac{pK_\mathrm{P}}{K_\mathrm{P} + pK_\mathrm{D}} \xrightarrow{p\to\infty} \frac{K_\mathrm{P}}{K_\mathrm{D}}$$

so, as $p \to \infty$, z tends to a constant, so we get a lead controller.

We use lead controllers as dynamic compensators for approximate PD control.

Double Integrator & Lead Compensator



To keep things simple, let's set $K_{\mathrm{P}} = K_{\mathrm{D}}$. Then:

$$K = K_{\mathrm{P}} + pK_{\mathrm{D}} = (1+p)K_{\mathrm{D}}$$

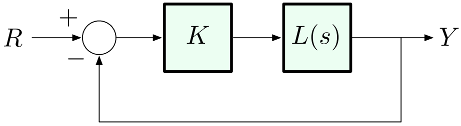
$$z = \frac{pK_{\mathrm{P}}}{K_{\mathrm{P}} + pK_{\mathrm{D}}} = \frac{pK_{\mathrm{D}}}{(1+p)K_{\mathrm{D}}} = \frac{p}{1+p} \xrightarrow{p \to \infty} 1$$

Since we can choose p and z directly, let's take

$$z = 1 \qquad \text{and} \qquad p \text{ large.}$$

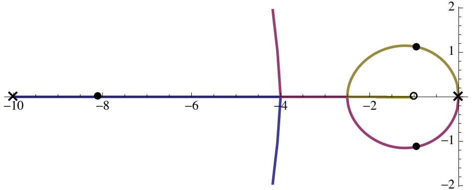
We expect to get behavior similar to PD control.

Double Integrator & Lead Compensator



$$L(s) = \frac{s+z}{s+p} \cdot \frac{1}{s^2} \overset{z=1}{=} \frac{s+1}{s^2(s+p)}$$

Let's try a few values of p. Here's $p = 10$:

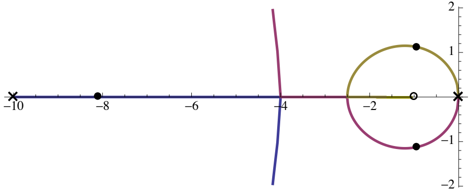


Close to $j\omega$-axis, this root locus looks similar to the PD root locus. However, the pole at $s = -10$ makes the locus look different for s far into LHP.

Double Integrator & Lead Compensator

$$L(s) = \frac{s+1}{s^2(s+p)}$$

Root locus for $p = 10$:



The design seems to look good: nice damping, can meet reasonable specs.

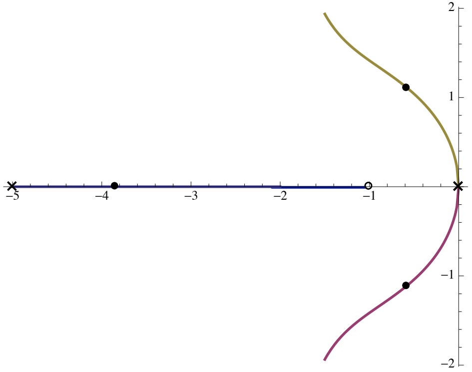
Any concerns with large values of p?

When p is large, we are very close to PD control, so we run into the same issue: noise amplification.
(This is just intuition for now — we will confirm it later using frequency-domain methods.)

Double Integrator & Lead Compensator

$$L(s) = \frac{s+1}{s^2(s+p)}$$

Let's try $p = 5$:

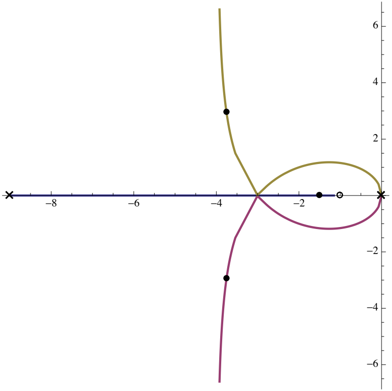


— for this value of p, the root locus is different, not nearly as nicely damped as for $p = 10$.

Double Integrator & Lead Compensator

$$L(s) = \frac{s+1}{s^2(s+p)}$$

Let's try p in between $p = 5$ and $p = 10$, say $p = 9$:

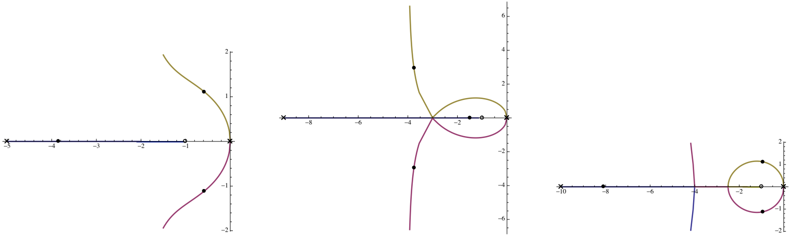


— for this value of p, the branches meet (*break in*) and separate (*break away*) at the same point on the real axis.

Summary on Design Trade-offs

From what we have seen so far:

- p large — good damping, but bad noise suppression (too close to PD); the branches first break in (meet at the real axis), then break away.
- p small — noise suppression is better, but RL is too close to $j\omega$-axis, which is not good; no break-in for small values of p.
- intermediate values of p — transition between two types of RL; break-in and break-away points are the same.



Lead Controller Design

With a lead controller in place, we have

$$KL(s) = K\frac{s+z}{s+p} \cdot G_p(s)$$

where the lead zero parameter z and lead pole parameter p are constrained to satisfy $z < p$.

In our example with $G_p(s) = 1/s^2$, we have set $z = 1$ to approximate PD control. Then $p > 1$ is our design parameter (and, of course, K is the gain parameter in the root locus).

Alternatively, we can assume that p is given (say, from noise suppression considerations), and we look for z that will give us a desired pole on the RL.

Is there a systematic procedure for doing this?

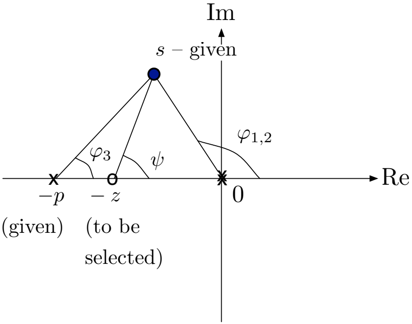
Pole Placement Using RL

Back to our example: double integrator with lead compensation

$$KL(s) = K\frac{s+z}{s+p} \cdot \frac{1}{s^2}$$

Problem: given p and a desired closed-loop pole s, find the value of z that will guarantee this (if possible).

Solution: use the phase condition

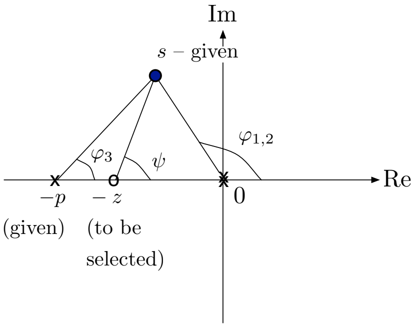


Must have

$$\underbrace{\psi}_{\text{angle from } s \text{ to zero}} - \sum_i \underbrace{\varphi_i}_{\text{angles from } s \text{ to poles}} = 180^\circ$$

So, we want $\psi = 180^\circ + \sum_i \varphi_i$

Pole Placement Using RL



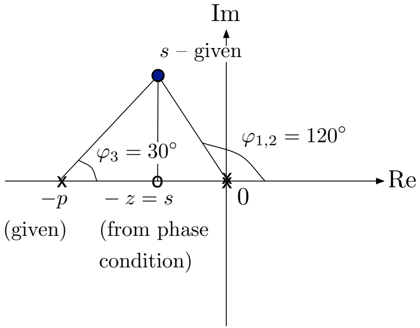
Suppose
$\varphi_1 = \varphi_2 = 120°$,
$\varphi_3 = 30°$.

We want $\psi = 180° + \sum_i \varphi_i$

Must have

$$\psi = 180° + 120° + 120° + 30°$$
$$= 450°$$
$$= 90° \mod 360°$$

Thus, we should have $z = -s$

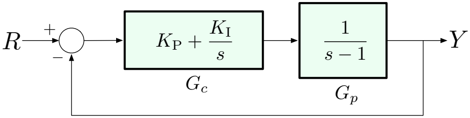


Control Design Using Root Locus

Case study: plant transfer function $G_p(s) = \dfrac{1}{s-1}$

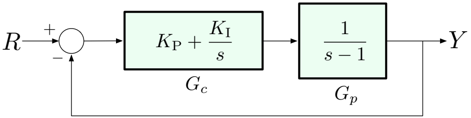
Control objective: stability and constant reference tracking

In earlier lectures, we saw that for perfect steady-state tracking we need PI control



Closed-loop poles are determined by:

$$1 + \left(K_P + \frac{K_I}{s} \right) \left(\frac{1}{s-1} \right) = 0$$



$$\text{Characteristic equation:} \quad 1 + \underbrace{\left(K_\mathrm{P} + \frac{K_\mathrm{I}}{s}\right)}_{G_c(s)} \underbrace{\left(\frac{1}{s-1}\right)}_{G_p(s)} = 0$$

To use the RL method, we need to convert it into the Evans form $1 + KL(s) = 0$, where $L(s) = \dfrac{b(s)}{a(s)} = \dfrac{s^m + b_1 s^{m-1} + \ldots}{s^n + a_1 s^{n-1} + \ldots}$

$$1 + \left(K_\mathrm{P} + \frac{K_\mathrm{I}}{s}\right) \frac{1}{s-1} = 1 + \frac{K_\mathrm{P} s + K_\mathrm{I}}{s} \frac{1}{s-1}$$

$$= 1 + K_\mathrm{P} \frac{s + K_\mathrm{I}/K_\mathrm{P}}{s(s-1)}$$

$$\implies K = K_\mathrm{P}, \; L(s) = \frac{s + K_\mathrm{I}/K_\mathrm{P}}{s(s-1)} \qquad \text{(assume } K_\mathrm{I}/K_\mathrm{P} \text{ fixed, } = 1)$$

Root Locus

$$L(s) = \frac{s+1}{s(s-1)}$$

Rule A: 2 branches

Rule B: branches start at $p_1 = 0, p_2 = 1$ (RHP!!)

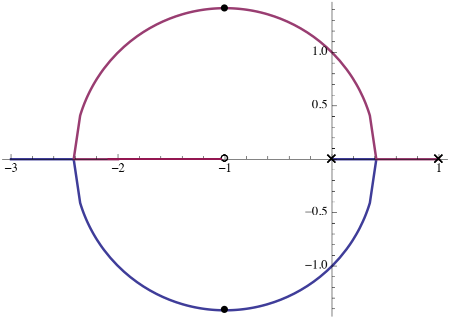
Rule C: branches end at $z_1 = -1, \pm\infty$

Rule D: real locus = $[0, 1], (-\infty, -1]$

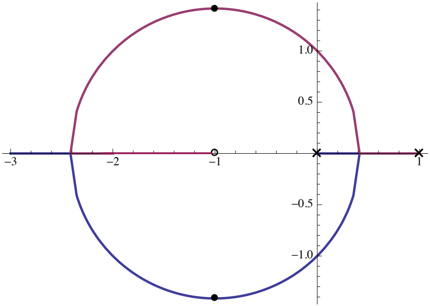
Rule E: asymptote at 180°

Rule F: $j\omega$-crossings:

$$a(s) + Kb(s) = 0$$
$$s(s-1) + K(s+1) = 0$$
$$s^2 + (K-1)s + K = 0$$
$$K_{\text{critical}} = 1 \implies \omega_0 = 1$$



Root Locus for PI Compensation



- The system is stable for $K > 1$ (from Routh-Hurwitz)
- For very large K, we get a completely damped system, with *negative real poles*
- Perfect steady-state tracking of constant references:

$$\frac{E}{R} = \frac{1}{1 + G_c G_p} = \frac{s(s-1)}{s(s-1) + K(s+1)}$$

DC gain$(R \to E) = 0$ (for $K > 1$)

- However: $1/s$ is not a stable element.

Approximate PI via Dynamic Compensation

PI control achieves the objective of stabilization and perfect steady-state tracking of constant references; however, just as with PD earlier, we want a *stable controller*.

Here's an idea:

$$\text{replace } K\frac{s+1}{s} \quad \text{by} \quad K\frac{s+1}{s+p}, \text{ where } p \text{ is small}$$

More generally, if $z = K_\mathrm{I}/K_\mathrm{P}$, then

$$\text{replace } K\frac{s+z}{s} \quad \text{by} \quad K\frac{s+z}{s+p}, \text{ where } p < z$$

This is lag compensation (or lag control)!

We use lag controllers as dynamic compensators for approximate PI control.