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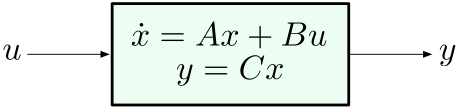
Reading: FPE, Chapter 7

State-Space Realizations



$$\downarrow$$

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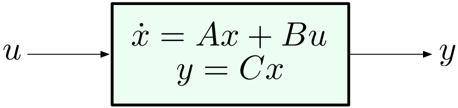

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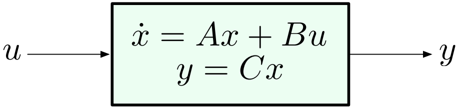

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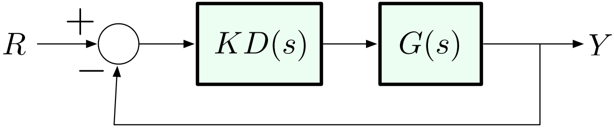
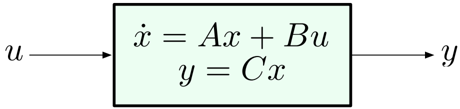
Then we add a controller to move the poles to desired locations:



Goal: Pole Placement by State Feedback

Consider a single-input system in state-space form:



Today, our goal is to establish the following fact:

If the above system is *controllable*, then we can assign arbitrary closed-loop poles by means of a state feedback law

$$
\begin{aligned}
u = -Kx &= -\begin{pmatrix} k_1 & k_2 & \ldots & k_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\
&= -(k_1 x_1 + \ldots + k_n x_n),
\end{aligned}
$$

where K is a $1 \times n$ matrix of feedback gains.

Review: Controllability

Consider a single-input system ($u \in \mathbb{R}$):

$$\dot{x} = Ax + Bu, \qquad y = Cx \qquad\qquad x \in \mathbb{R}^n$$

The Controllability Matrix is defined as

$$\mathcal{C}(A, B) = \left[B \,|\, AB \,|\, A^2B \,|\, \dots \,|\, A^{n-1}B \right]$$

We say that the above system is controllable if its controllability matrix $\mathcal{C}(A, B)$ is *invertible*.

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- As we will see today, if the system is controllable, then we may assign arbitrary closed-loop poles by *state feedback* of the form $u = -Kx$.

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- As we will see today, if the system is controllable, then we may assign arbitrary closed-loop poles by *state feedback* of the form $u = -Kx$.
- Whether or not the system is controllable depends on its state-space realization.

Controller Canonical Form

A single-input state-space model

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

is said to be in Controller Canonical Form (CCF) is the matrices A, B are of the form

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ * & * & * & \dots & * & * \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

A system in CCF is *always controllable*!!

(The proof of this for $n > 2$ uses the Jordan canonical form, we will not worry about this.)

Coordinate Transformations

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- Hence, we need a way of constructing a CCF state-space realization of a given controllable system.
- We will do this by suitably changing the coordinate system for the state vector.

Coordinate Transformations and State-Space Models

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad \xrightarrow{\quad T \quad} \quad \begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x} \end{aligned}$$

where $\bar{A} = TAT^{-1}, \qquad \bar{B} = TB, \qquad \bar{C} = CT^{-1}$

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This gives us a way of systematically passing to CCF.

Example: Converting a Controllable System to CCF

$$A = \begin{pmatrix} -15 & 8 \\ -15 & 7 \end{pmatrix}, \; B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad (C \text{ is immaterial})$$

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Step 3: Compute T.

$$T = \mathcal{C}(\bar{A}, \bar{B}) \cdot [\mathcal{C}(A, B)]^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -8 \end{pmatrix} \begin{pmatrix} 8 & -7 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Finally, Pole Placement via State Feedback

Consider a state-space model

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Let's introduce a *state feedback law*

$$u = -Ky \equiv -Kx$$
$$= -\begin{pmatrix} k_1 & k_2 & \dots & k_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = -(k_1 x_1 + \dots + k_n x_n)$$

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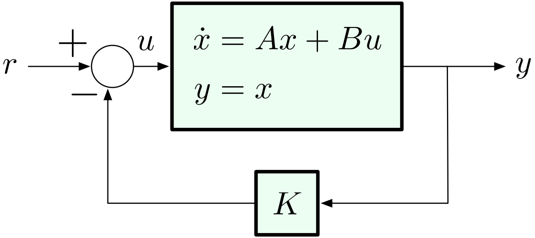
Closed-loop system:

$$\dot{x} = Ax - BKx = (A - BK)x$$
$$y = x$$

Pole Placement via State Feedback

Let's also add a reference input:



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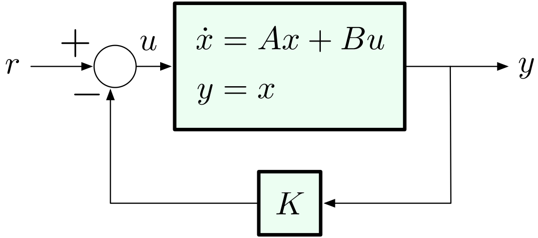

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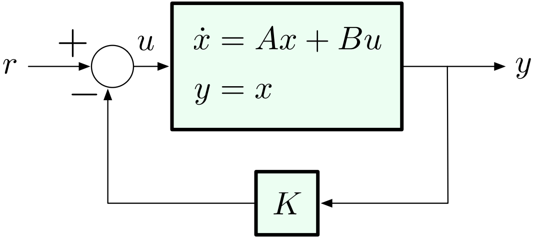

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Take the Laplace transform:

$$sX(s) = (A - BK)X(s) + BR(s), \;\; Y(s) = X(s)$$

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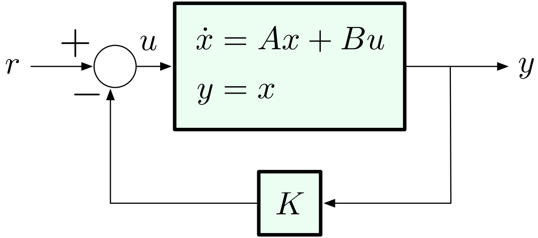

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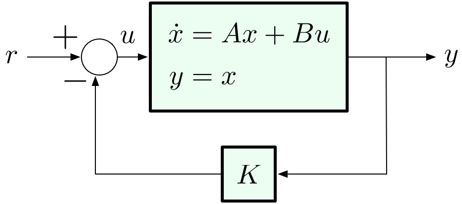

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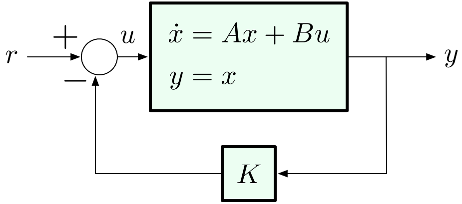

assigning closed-loop poles = assigning eigenvalues of $A - BK$

Now we will see that this is particularly straightforward if the (A, B) system is in CCF.

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_2 & -a_1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

The Beauty of CCF

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Claim.

$$\det(Is - A) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n$$

— the last row of the A matrix in CCF consists of the coefficients of the characteristic polynomial, in reverse order, with "−" signs.

Proof of the Claim

A nice way is via Laplace transforms:

$$\dot{x} = Ax + Bu$$

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$$\vdots \qquad\qquad \vdots$$

$$\dot{x}_n = -\sum_{i=1}^{n} a_{n-i+1} x_i + u \qquad\qquad \underbrace{\left(s^n + a_1 s^{n-1} + \ldots + a_n\right)}_{\text{char. poly.}} X_1 = U$$

... And, Back to Pole Placement

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— still in CCF!!

Pole Placement in CCF

$$\dot{x} = (A - BK)x + Br, \qquad y = Cx$$

$$A - BK = -\begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ a_n + k_1 & a_{n-1} + k_2 & \dots & a_2 + k_{n-1} & a_1 + k_n \end{pmatrix}$$

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Closed-loop poles are the roots of the characteristic polynomial

$$\det(Is - A + BK)$$
$$= s^n + (a_1 + k_n)s^{n-1} + \ldots + (a_{n-1} + k_2)s + (a_n + k_1)$$

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Hence the name Controller Canonical Form — convenient for control design.

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3. Convert back to original coordinates.

Example

Given $\dot{x} = Ax + Bu$

$$A = \begin{pmatrix} -15 & 8 \\ -7 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Goal: apply state feedback to place closed-loop poles at $-10 \pm j$.

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$$T = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad \longrightarrow \qquad \bar{A} = \begin{pmatrix} 0 & 1 \\ -15 & -8 \end{pmatrix}, \ \bar{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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This gives the control law

$$u = -\bar{K}\bar{x} = -\begin{pmatrix} 86 & 12 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}$$

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The desired state feedback law is

$$u = \begin{pmatrix} -86 & 74 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$