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- Review: state-space notions: canonical forms, controllability.
- Today's topic: controllability, stability, and pole-zero cancellations; effect of coordinate transformations; conversion of any controllable system to CCF.

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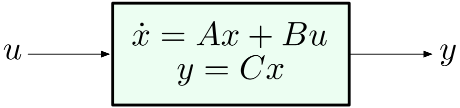
Reading: FPE, Chapter 7

State-Space Realizations



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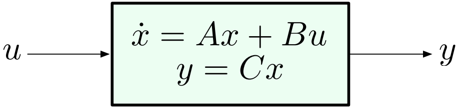

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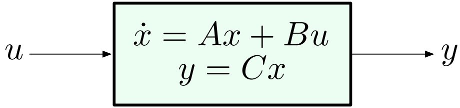

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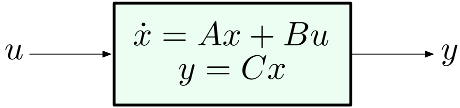

- a given transfer function $G(s)$ can be realized using infinitely many state-space models
- certain properties make some realizations preferable to others
- one such property is *controllability*

Controllability Matrix

Consider a single-input system ($u \in \mathbb{R}$):

$$\dot{x} = Ax + Bu, \qquad y = Cx \qquad\qquad x \in \mathbb{R}^n$$

The Controllability Matrix is defined as

$$\mathcal{C}(A, B) = \left[B \,|\, AB \,|\, A^2 B \,|\, \ldots \,|\, A^{n-1} B \right]$$

We say that the above system is controllable if its controllability matrix $\mathcal{C}(A, B)$ is *invertible*.

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- As we will see later, if the system is controllable, then we may assign arbitrary closed-loop poles by *state feedback* of the form $u = -Kx$.
- Whether or not the system is controllable depends on its state-space realization.

Example: Computing $\mathcal{C}(A, B)$

Let's get back to our old friend:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \qquad y = \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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$$\det \mathcal{C} = -1 \neq 0 \qquad \implies \qquad \text{system is controllable}$$

Controller Canonical Form

A single-input state-space model

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

is said to be in Controller Canonical Form (CCF) is the matrices A, B are of the form

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ * & * & * & \dots & * & * \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

A system in CCF is *always controllable*!!

(The proof of this for $n > 2$ uses the Jordan canonical form, we will not worry about this.)

CCF with Arbitrary Zeros

In our example, we had $G(s) = \dfrac{s+1}{s^2+5s+6}$, with a minimum-phase zero at $z = -1$.

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A system in CCF is controllable for any locations of the zeros.

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Start with the CCF

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Convert to OCF: $\qquad (A \mapsto A^T, B \mapsto C^T, C \mapsto B^T)$

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But is it *controllable*?

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The OCF realization of the transfer function $G(s) = \dfrac{s-z}{s^2+5s+6}$ is not controllable when $z = -2$ or -3, even though the CCF is always controllable.

Beware of Pole-Zero Cancellations!

The OCF realization of the transfer function

$$G(s) = \frac{s - z}{s^2 + 5s + 6}$$

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For $z = -2$, $G(s)$ is a first-order transfer function, which can always be realized by this 1st-order controllable model:

$$\dot{x}_1 = -3x_1 + u, \; y = x_1 \quad \longrightarrow \quad G(s) = \frac{1}{s + 3}$$

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or a noncontrollable two-dimensional state-space model

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} u, \qquad y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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Thus, even the *state dimension* of a realization of a given t.f. is not unique!!

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The transfer function can mask undesirable internal state behavior!!

Pole-Zero Cancellations and Stability

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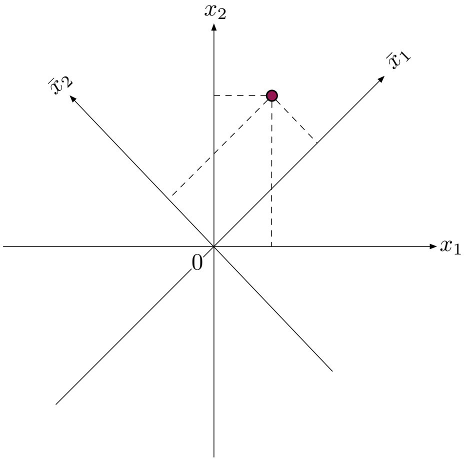
This is equivalent to having no RHP open-loop poles and no pole-zero cancellations in RHP.

Coordinate Transformations

Now that we have seen that a given transfer function can have many different state-space realizations, we would like a systematic procedure of generating such realizations, preferably with favorable properties (like controllability).

One such procedure is by means of *coordinate transformations*.

Coordinate Transformations



$x \longmapsto \bar{x} = Tx,$ $\qquad T \in \mathbb{R}^{n\times n}$nonsingular

$x = T^{-1}\bar{x}$ $\qquad$ (go back and forth between the coordinate systems)

Coordinate Transformations

For example,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$$

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The transformation is invertible: $\det T = -2$, and

$$T^{-1} = \frac{1}{\det T} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

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Or we can see this directly:

$$\bar{x}_1 + \bar{x}_2 = 2x_1; \quad \bar{x}_1 - \bar{x}_2 = 2x_2$$

Coordinate Transformations and State-Space Models

Consider a state-space model

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

and a change of coordinates $\bar{x} = Tx$ (T invertible).

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What happens to

- the transfer function?
- the controllability matrix?

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Claim: The transfer function doesn't change.

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Since $\det T \neq 0$, $\det \mathcal{C}(\bar{A}, \bar{B}) \neq 0$ if and only if $\det \mathcal{C}(A, B) \neq 0$.

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Thus, the new system is controllable if and only if the old one is.

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This is a recipe for going from one *controllable* realization of a given t.f. to another.

CCF is the most convenient controllable realization of a given t.f., so we want to *convert a given controllable system to CCF* (useful for control design).

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$\det \mathcal{C} = -1$ – controllable

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so we need to find the coefficients a_1, a_2.

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Therefore, the new controllability matrix should be

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In the next lecture, we will see why CCF is so useful.