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- Review: frequency-domain design method.
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Reading: FPE, Chapter 7

Frequency-Domain vs. State-Space

- 90% of industrial controllers are designed using frequency-domain methods (PID is a popular architecture)
- 90% of current research in systems and control is in the state-space framework

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To be able to talk to control engineers *and* follow progress in the field, we need to know both methods and *understand the connections between them.*

State-Space Methods

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- the mathematics is different: heavy use of *linear algebra*
- this is just a short introduction; to learn this material properly, take ECE 515

A General State-Space Model

$$\text{state } x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \qquad \text{input } u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \in \mathbb{R}^m$$

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where:

A – system matrix ($n \times n$)

B – input matrix ($n \times m$)

C – output matrix ($p \times n$)

D – feedthrough matrix ($p \times m$)

From State-Space to Transfer Function

Let us find the *transfer function* from u to y corresponding to the state-space model

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- in the scalar case ($x, y, u \in \mathbb{R}$), we took the Laplace transform
- the same idea here when working with vectors: just do it component by component

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Recall matrix-vector multiplication:

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$$y_\ell = \sum_{j=1}^{n} c_{\ell j} x_j + \sum_{k=1}^{m} d_{\ell k} u_k$$

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To find the input-output t.f., set the IC to 0:

$$Y(s) = G(s)U(s), \qquad \text{where } G(s) = C(Is - A)^{-1}B + D$$

From State-Space to Transfer Function

The transfer function from u to y, corresponding to

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is given by

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Observe that $G(s)$ contains information about the state-space matrices A, B, C, D!!

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Important!!

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  and its roots are the eigenvalues of A
- G is (open-loop) stable if all eigenvalues of A lie in LHP.

Example: Computing $G(s)$

Consider the state-space model in Controller Canonical Form (CCF)*:

* We will explain this terminology later.

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$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \qquad y = \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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— this is a *single-input, single-output* (SISO) system, since $u, y \in \mathbb{R}$; the state is two-dimensional.

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A useful formula for the inverse of a 2×2 matrix:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \det M \neq 0 \quad \Longrightarrow \quad M^{-1} = \frac{1}{\det M} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

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Applying the formula, we get

$$\begin{aligned} (Is - A)^{-1} &= \frac{1}{\det(Is - A)} \begin{pmatrix} s+5 & 1 \\ -6 & s \end{pmatrix} \\ &= \frac{1}{s^2 + 5s + 6} \begin{pmatrix} s+5 & 1 \\ -6 & s \end{pmatrix} \end{aligned}$$

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- note how coefficients 5 and 6 appear in both $G(s)$ and A!!

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Answer: There are infinitely many!

State-Space Realizations of Transfer Functions

Start with

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and consider a new state-space model

$$\dot{x} = \bar{A}x + \bar{B}u, \qquad y = \bar{C}x$$

with

$$\bar{A} = A^T = \begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix}, \quad \bar{B} = C^T = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \bar{C} = B^T = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

This is a different state-space model!

State-Space Realizations of Transfer Functions

Claim: The state-space model

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But the state-space model is now in the Observer Canonical Form (OCF):

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -6 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u, \qquad y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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Yet another realization of $G(s) = \dfrac{s+1}{s^2+5s+6}$ can be extracted from the partial-fractions decomposition:

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This is the Modal Canonical Form (MCF):

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u, \qquad y = \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

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- certain properties make some realizations preferable to others
- one such property is *controllability*

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$$\dot{x} = Ax + Bu, \qquad y = Cx \qquad\qquad x \in \mathbb{R}^n$$

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We say that the above system is controllable if its controllability matrix $\mathcal{C}(A, B)$ is *invertible*.

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(This definition is only true for the single-input case; the multiple-input case involves the *rank* of $\mathcal{C}(A, B)$.)

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- As we will see later, if the system is controllable, then we may assign arbitrary closed-loop poles by *state feedback* of the form $u = -Kx$.
- Whether or not the system is controllable depends on its state-space realization.

Example: Computing $\mathcal{C}(A, B)$

Let's get back to our old friend:

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$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B} u, \qquad y = \underbrace{\begin{pmatrix} 1 & 1 \end{pmatrix}}_{C} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Here, $x \in \mathbb{R}^2 \implies A \in \mathbb{R}^{2\times 2} \implies \mathcal{C}(A, B) \in \mathbb{R}^{2\times 2}$

$$\mathcal{C}(A, B) = [B \mid AB] \qquad AB = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

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$$\det \mathcal{C} = -1 \neq 0 \qquad \implies \qquad \text{system is controllable}$$

Controller Canonical Form

A single-input state-space model

$$\dot{x} = Ax + Bu, \qquad y = Cx$$

is said to be in Controller Canonical Form (CCF) is the matrices A, B are of the form

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ * & * & * & \dots & * & * \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

A system in CCF is *always controllable*!!

(The proof of this for $n > 2$ uses the Jordan canonical form, we will not worry about this.)

CCF with Arbitrary Zeros

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Since A, B are the same, $\mathcal{C}(A, B)$ is the same $\Longrightarrow$ the system is still controllable.

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A system in CCF is controllable for any locations of the zeros.