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- Today's topic: Nyquist stability criterion

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Reading: FPE, Chapter 6

Review: Frequency Domain Design Method

Design based on Bode plots is good for:

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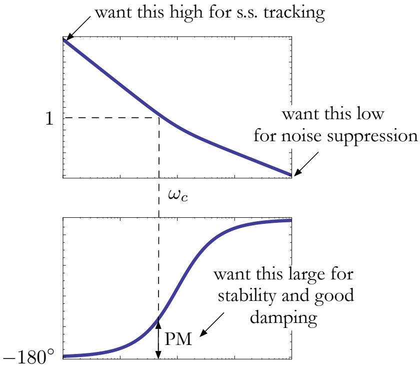

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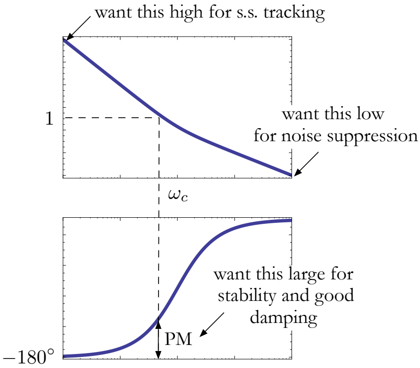

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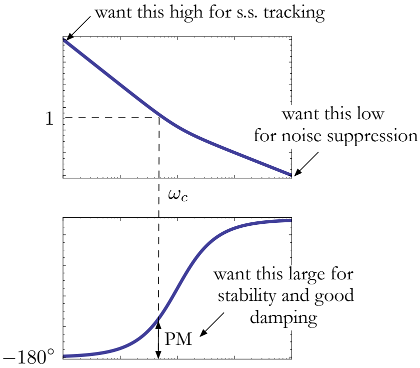

- evaluating the design and seeing which way to change it
- using experimental data (frequency response of the uncontrolled system can be measured experimentally)

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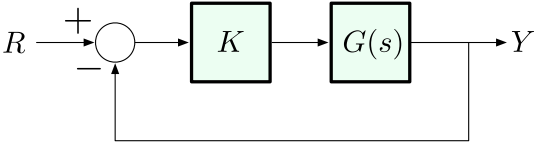
What we want is a frequency-domain substitute for the Routh–Hurwitz criterion — this is the Nyquist criterion, which we will discuss in today's lecture.

Nyquist Stability Criterion



Goal: count the number of RHP poles (if any) of the closed-loop transfer function

$$\frac{KG(s)}{1+KG(s)}$$

based on frequency-domain characteristics of the plant transfer function $G(s)$

Review: Nyquist Plot

Consider an arbitrary *strictly proper* transfer function H:

$$H(s) = \frac{(s - z_1)\dots(s - z_m)}{(s - p_1)\dots(s - p_n)}, \qquad m < n$$

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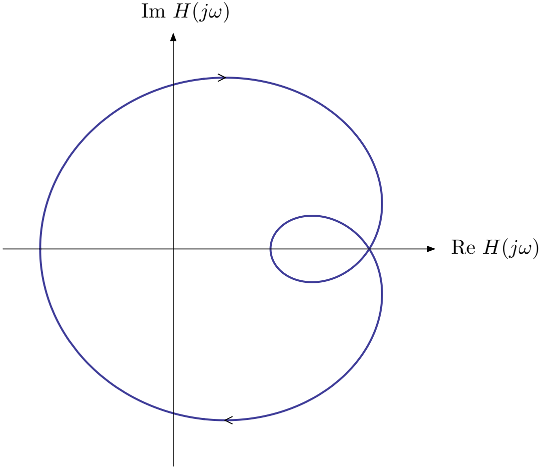
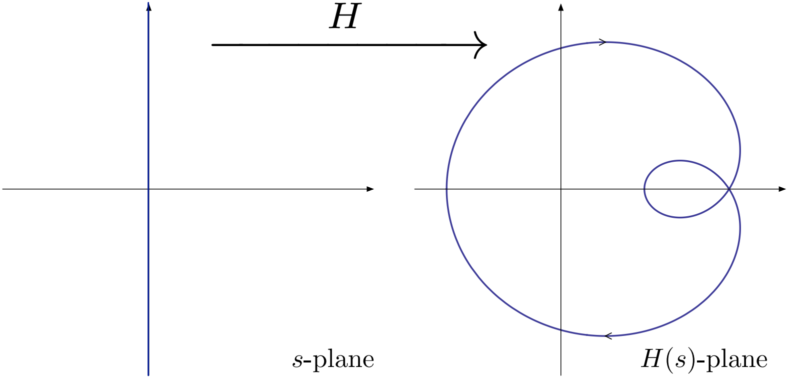
Nyquist plot: $\operatorname{Im} H(j\omega)$ vs. $\operatorname{Re} H(j\omega)$ as ω varies from $-\infty$ to ∞



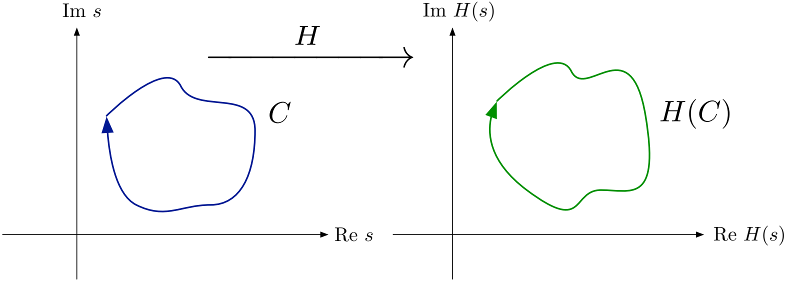
Nyquist Plot as a Mapping of the s-Plane

We can view the Nyquist plot of H as the image of the imaginary axis $\{j\omega : -\infty < \omega < \infty\}$ under the mapping $H : \mathbb{C} \to \mathbb{C}$



Transformation of a Closed Contour Under H

If we choose any closed curve (or *contour*) C on the left, it will get mapped by H to some other curve (contour) on the right:



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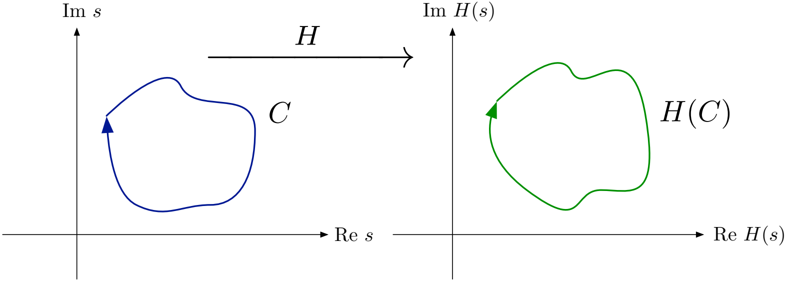

Important: when working with contours in the complex plane, always keep track of the direction in which we traverse the contour (clockwise vs. counterclockwise)!!

Phase of H Along a Contour

For any $s \in \mathbb{C}$, the phase (or *argument*) of $H(s)$ is

$$\begin{aligned} \angle H(s) &= \angle \frac{(s-z_1)\dots(s-z_m)}{(s-p_1)\dots(s-p_n)} \\ &= \sum_{i=1}^{m} \angle(s-z_i) - \sum_{j=1}^{n} \angle(s-p_j) \\ &= \sum_{i=1}^{m} \psi_i - \sum_{j=1}^{n} \varphi_j \end{aligned}$$

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We are interested in how $\angle H(s)$ changes as s traverses a closed, clockwise (↻) oriented contour C in the complex plane.

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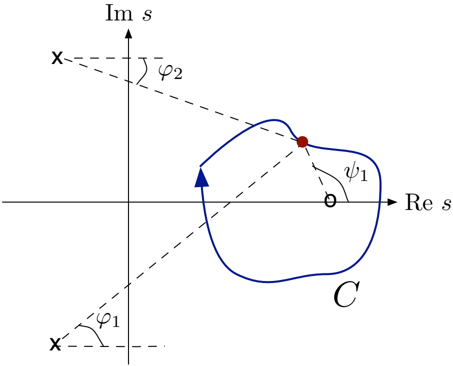
We will look at several cases, depending on how the contour is located relative to poles and zeros of H.

Case 1: Contour Encircles a Zero

Suppose that C is a closed, ↻-oriented contour in $\mathbb{C}$ that encircles a zero of $H(s)$:



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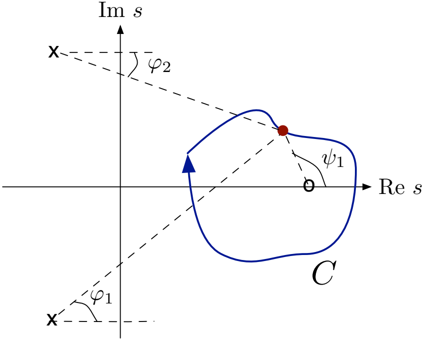

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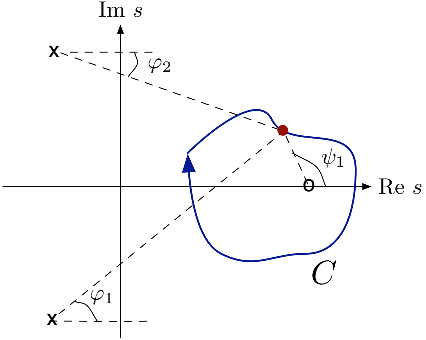

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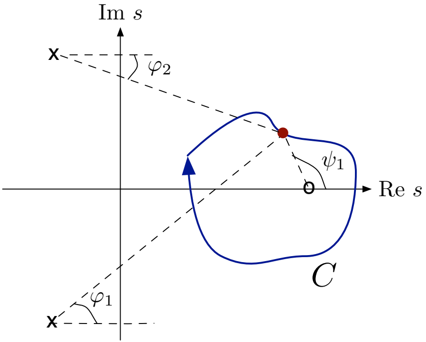

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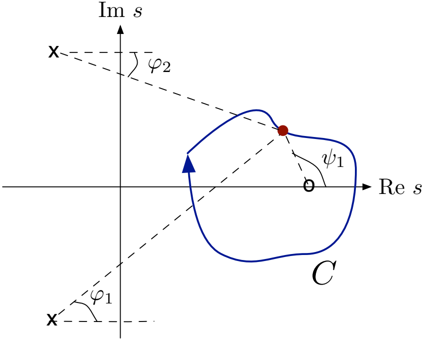

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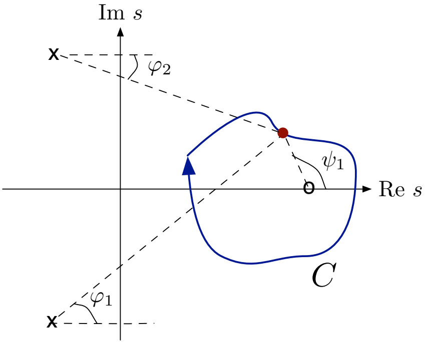

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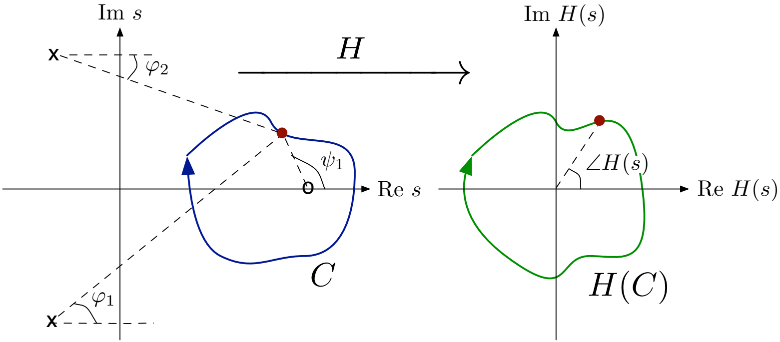

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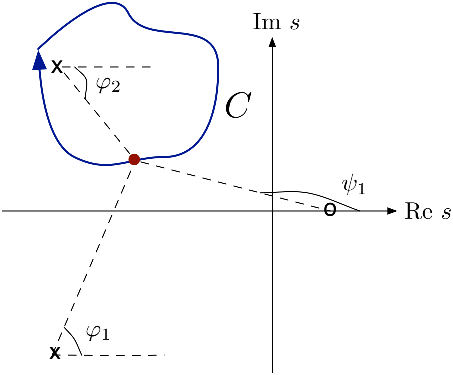

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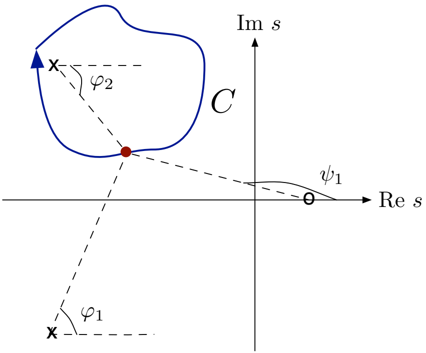

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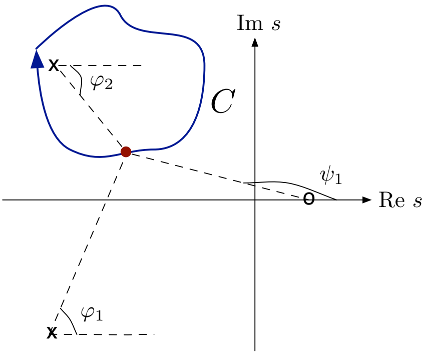

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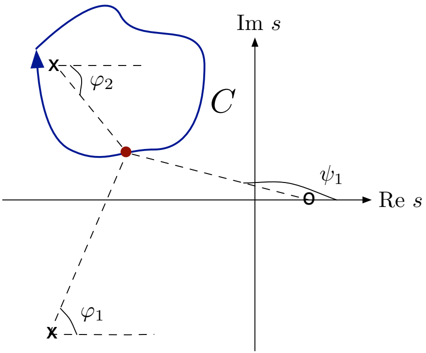

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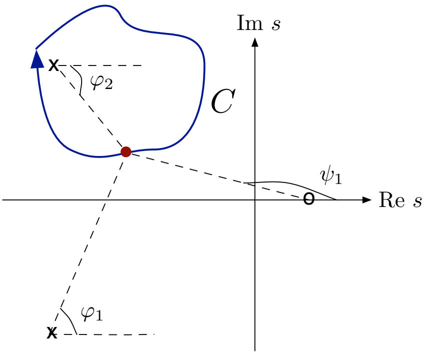

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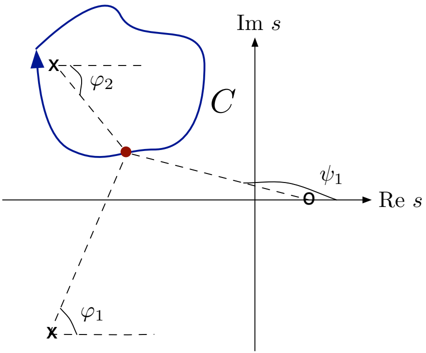

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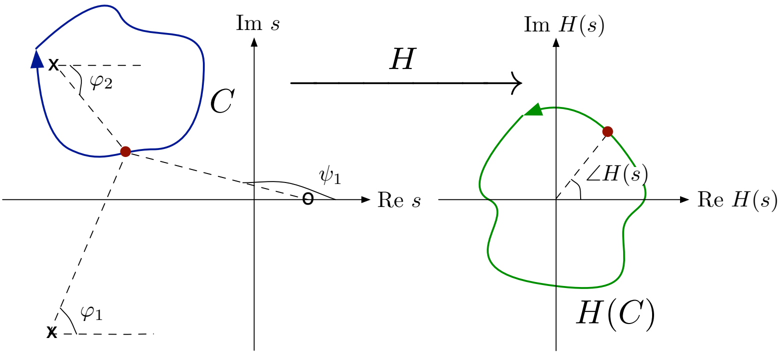

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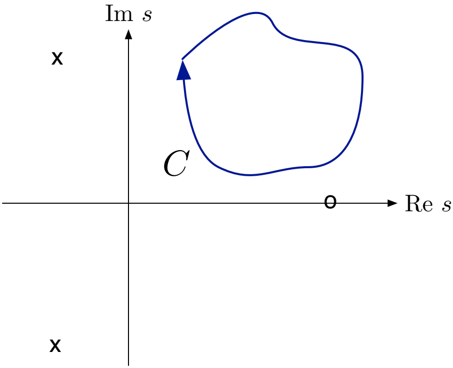

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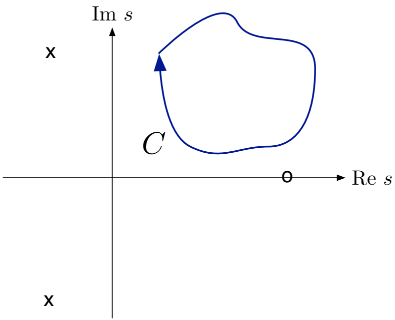

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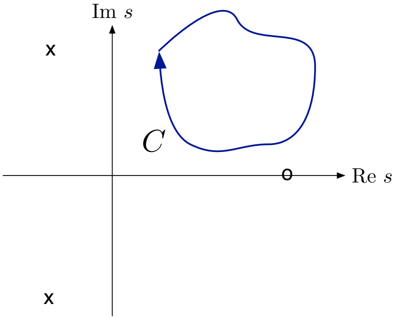

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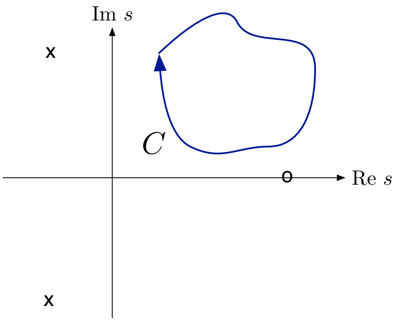

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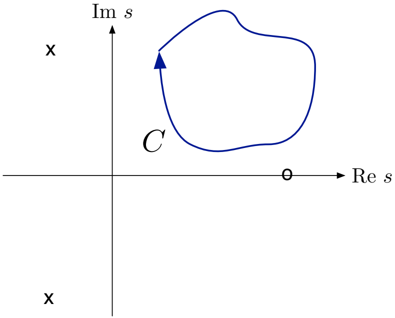

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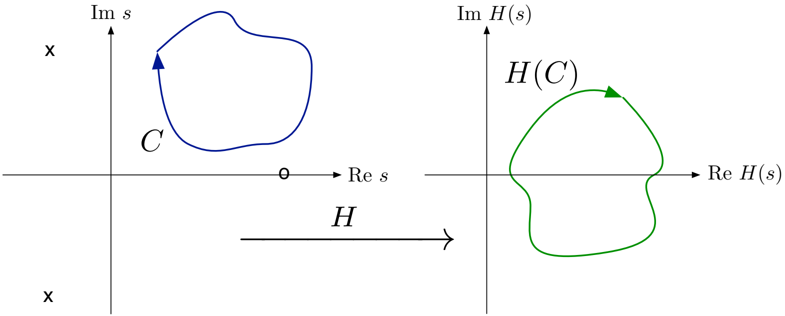

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The Argument Principle

These special cases all lead to the following general result:

The Argument Principle. Let C be a closed, clockwise ↻ oriented contour not passing through any zeros or poles* of $H(s)$. Let $H(C)$ be the image of C under the map $s \mapsto H(s)$:

$$H(C) = \{H(s) : s \in \mathbb{C}\}.$$

Then:

$$\begin{aligned}&\#(\text{clockwise encirclements} \circlearrowright \text{ of } 0 \text{ by } H(C)) \\ &= \#(\text{zeros of } H(s) \text{ inside } C) - \#(\text{poles of } H(S) \text{ inside } C).\end{aligned}$$

More succinctly,

$$N = Z - P$$

* will see the reason for this later ...

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The Argument Principle

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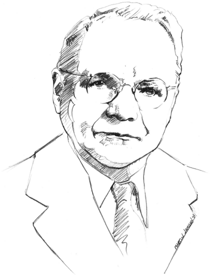
- If $N < 0$, it means that $H(C)$ encircles the origin counterclockwise ($\circlearrowleft$).
- We do not want C to pass through any pole of H because then $H(C)$ would not be defined.
- We also do not want C to pass through any zero of H because then $0 \in H(C)$, so #(encirclements) is not well-defined.

From Argument Principle to Nyquist Criterion



Harry Nyquist
(1889–1976)

- We are interested in RHP poles, so let's choose a suitable contour C that *encloses the RHP*:

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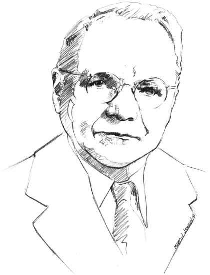

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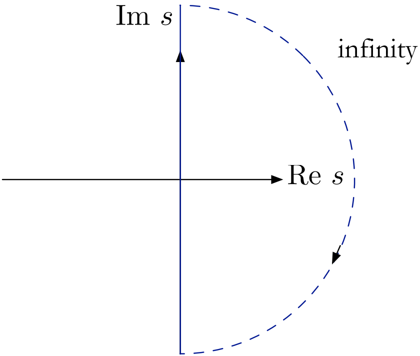

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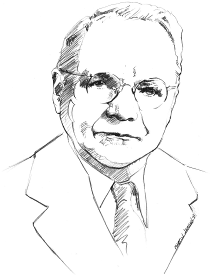

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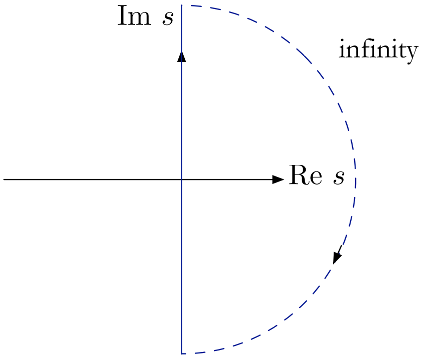

- From now on, C = imaginary axis plus the "path around infinity."

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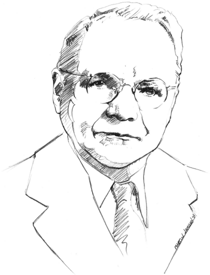

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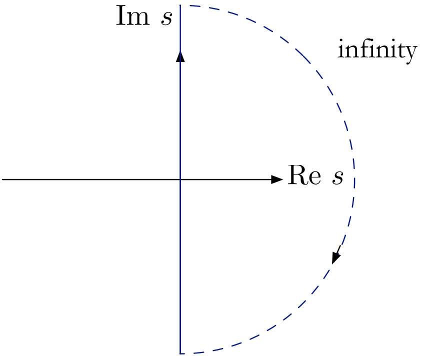
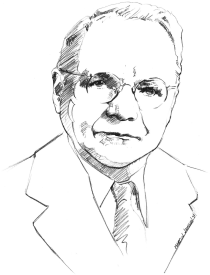


- From now on, C = imaginary axis plus the "path around infinity."
- If H is strictly proper, then $H(\infty) = 0$.

From Argument Principle to Nyquist Criterion



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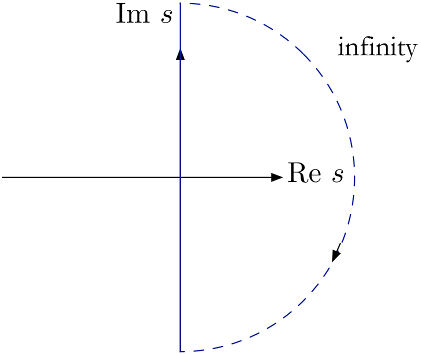
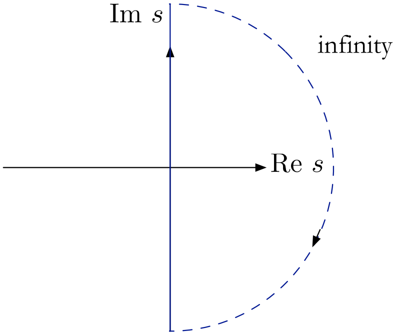


With this choice of C,

$$H(C) = \text{Nyquist plot of } H$$

(image of the imaginary axis under the map $H : \mathbb{C} \to \mathbb{C}$; if H is strictly proper, $0 = H(\infty)$)

From Argument Principle to Nyquist Criterion



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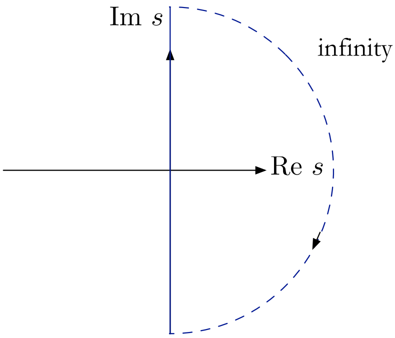

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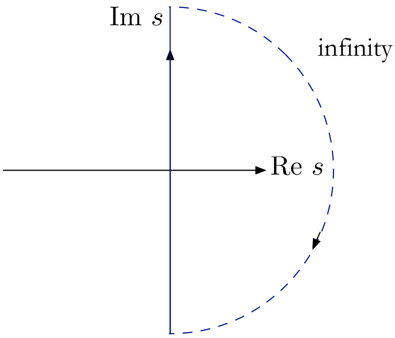
We are interested in RHP roots of $1 + KG(s)$, where G is the plant transfer function.

From Argument Principle to Nyquist Criterion



$$H(C) = \text{Nyquist plot of } H$$

We are interested in RHP roots of $1 + KG(s)$, where G is the plant transfer function.

Thus, we choose $H(s) = 1 + KG(s)$

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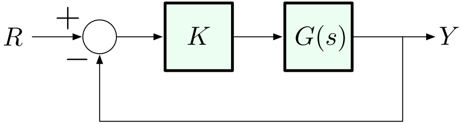

We now examine the Nyquist plot of $H(s) = 1 + KG(s)$.

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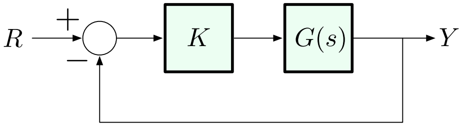

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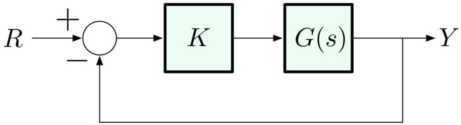
From Argument Principle to Nyquist Criterion



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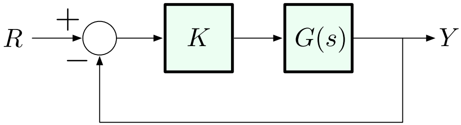

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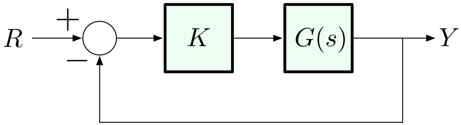

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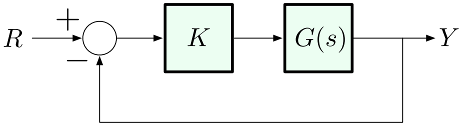

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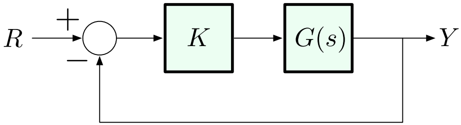

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Now we extract information about RHP roots of $1 + KG(s)$

Nyquist Criterion: N

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$$\begin{aligned} N &= \#(\circlearrowright \text{ encirclements of } 0 \text{ by Nyquist plot of } 1 + KG(s)) \\ &= \#(\circlearrowright \text{ encirclements of } {\color{blue}-1} \text{ by Nyquist plot of } {\color{blue}KG(s)}) \\ &= \#(\circlearrowright \text{ encirclements of } {\color{green}-1/K} \text{ by Nyquist plot of } {\color{green}G(s)}) \end{aligned}$$

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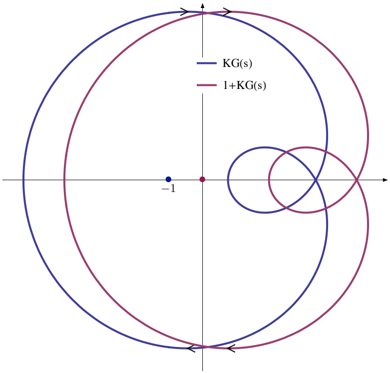

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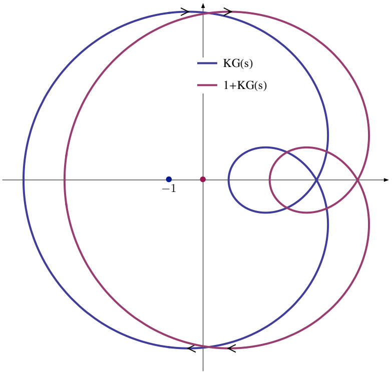
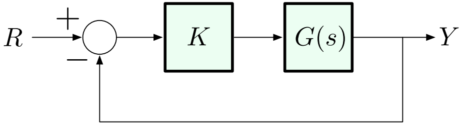


— can be read off the Nyquist plot of the *open-loop* t.f. G!!

Nyquist Criterion: Z



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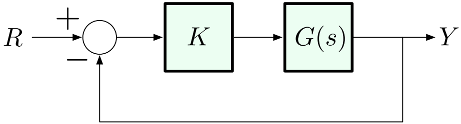

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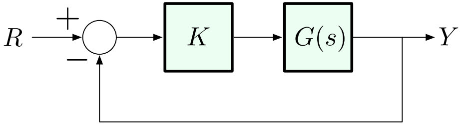

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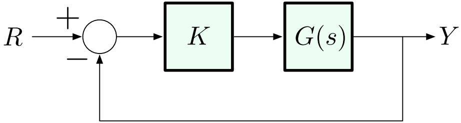

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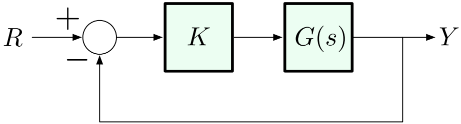

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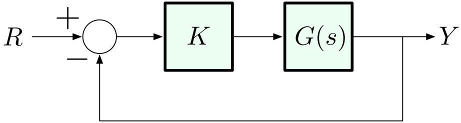

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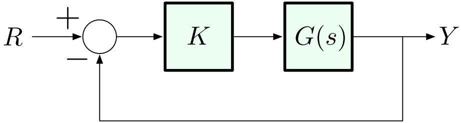

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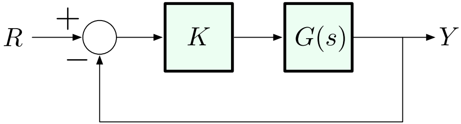

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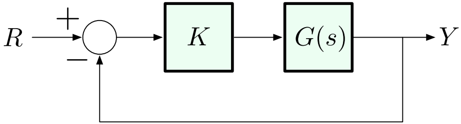

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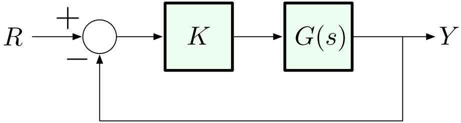

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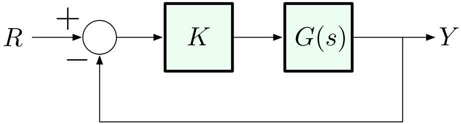

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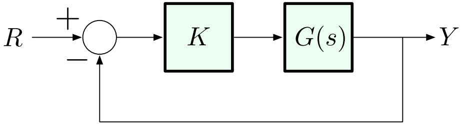

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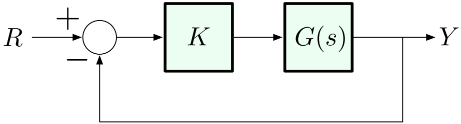

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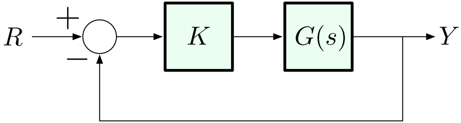

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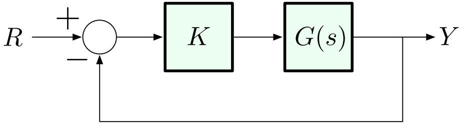
The Nyquist Theorem



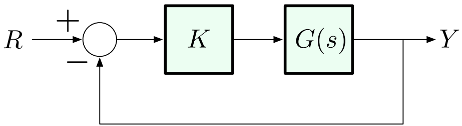
Nyquist Theorem (1928) Assume that $G(s)$ has no poles on the imaginary axis*, and that its Nyquist plot does not pass through the point $-1/K$. Then

$$N = Z - P$$

$$\#(\circlearrowright \text{ of } -1/K \text{ by Nyquist plot of } G(s))$$
$$= \#(\text{RHP closed-loop poles}) - \#(\text{RHP open-loop poles})$$

* Easy to fix: draw an infinitesimally small circular path that goes *around* the pole and stays in RHP

The Nyquist Stability Criterion



$$\underbrace{N}_{\#(\circlearrowright \text{ of } -1/K)} = \underbrace{Z}_{\#(\text{unstable CL poles})} - \underbrace{P}_{\#(\text{unstable OL poles})}$$

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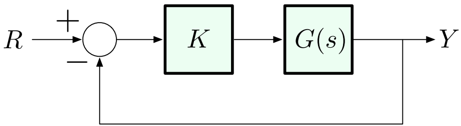

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Nyquist Stability Criterion. Under the assumptions of the Nyquist theorem, the closed-loop system (at a given gain K) is stable *if and only if* the Nyquist plot of $G(s)$ encircles the point $-1/K$ P times *counterclockwise*, where P is the number of unstable (RHP) open-loop poles of $G(s)$.

Applying the Nyquist Criterion

Workflow:

$$\text{Bode } M \text{ and } \phi\text{-plots} \quad \longrightarrow \quad \text{Nyquist plot}$$

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Advantages of Nyquist over Routh–Hurwitz

- can work directly with experimental frequency response data (e.g., if we have the Bode plot based on measurements, but do not know the transfer function)
- less computational, more geometric (came 55 years after Routh)

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$$G(s) = \frac{1}{(s+1)(s+2)}$$

(no open-loop RHP poles)

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Characteristic equation:

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We will now reproduce this answer using the Nyquist criterion.

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— Nyquist plots are always *symmetric w.r.t. the real axis*!!

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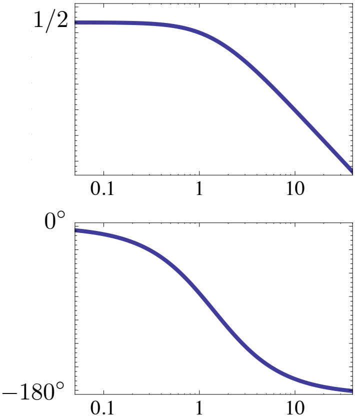

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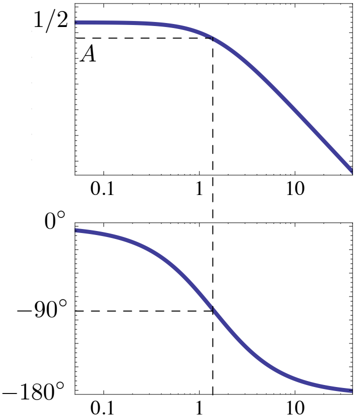

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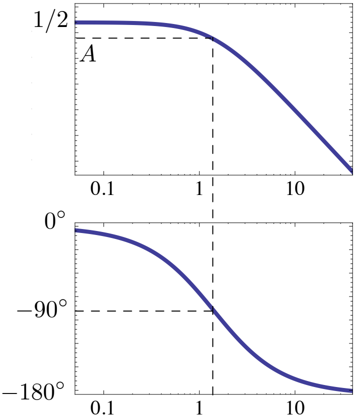
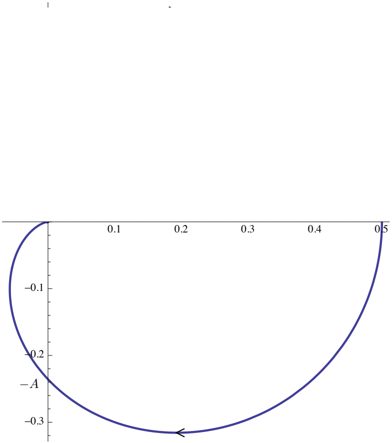


Nyquist plot:

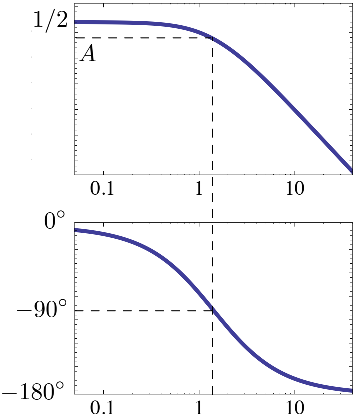


Example

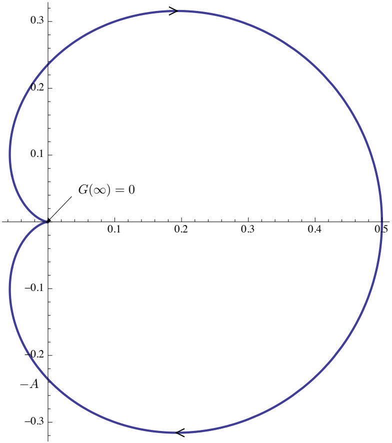
$$G(s) = \frac{1}{(s+1)(s+2)}$$

(no open-loop RHP poles)

Bode plot:



Nyquist plot:



Example: Applying the Nyquist Criterion

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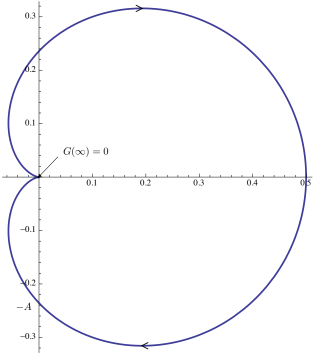

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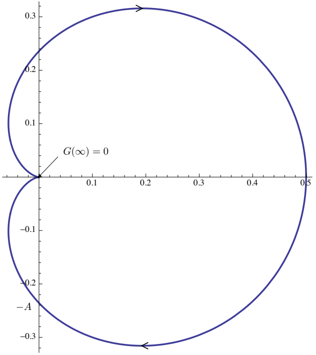

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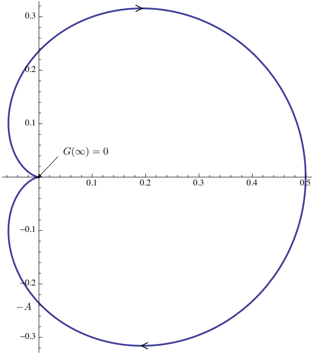

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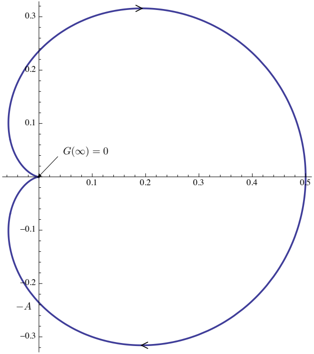

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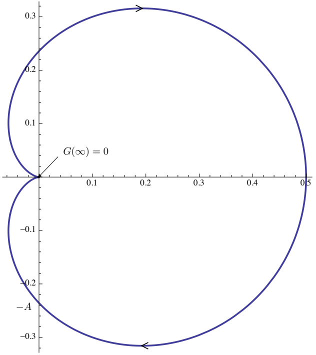

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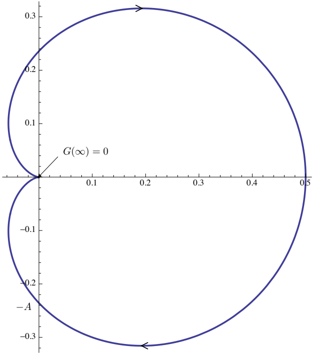

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