Plan of the Lecture

- Review: transient and steady-state response; DC gain and the FVT
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Reading: FPE, Sections 3.1–3.2; lab manual

System Modeling Diagrams

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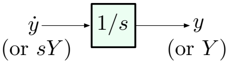
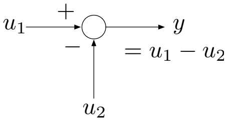
We will take smaller blocks from some given *library* and play with them to create/build more complicated systems.

All-Integrator Diagrams

Our library will consist of three building blocks:



integrator



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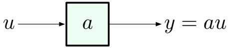

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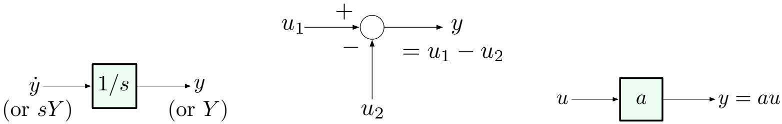

integrator | summing junction | constant gain

Two warnings:

- We can (and will) work either with u, y (time domain) or with U, Y (s-domain) — will often go back and forth
- When working with block diagrams, we typically ignore initial conditions.

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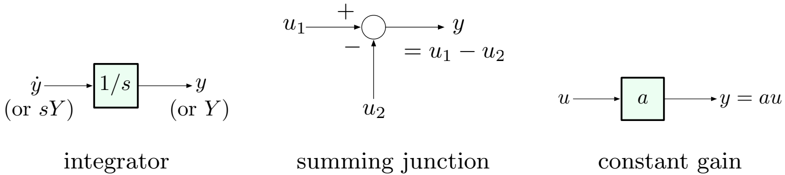

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- When working with block diagrams, we typically ignore initial conditions.

This is the *lowest level* we will go to in lectures; in the labs, you will implement these blocks using op amps.

Example 1

Build an all-integrator diagram for

$$\ddot{y} = u \qquad \Longleftrightarrow \qquad s^2 Y = U$$

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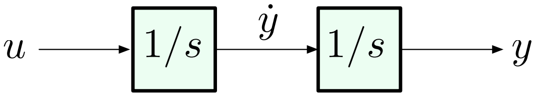
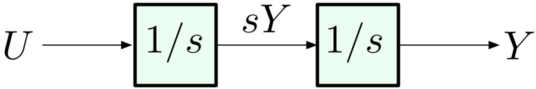


or



Example 2

(building on Example 1)

$$\ddot{y} + a_1\dot{y} + a_0 y = u \qquad \Longleftrightarrow \qquad s^2Y + a_1 sY + a_0 Y = U$$

$$\text{or} \qquad Y(s) = \frac{U(s)}{s^2 + a_1 s + a_0}$$

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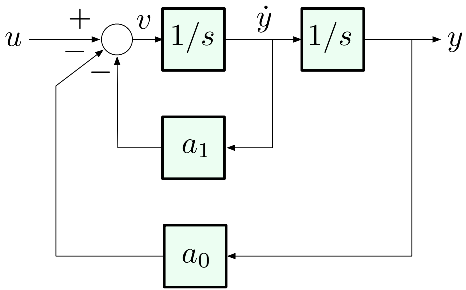

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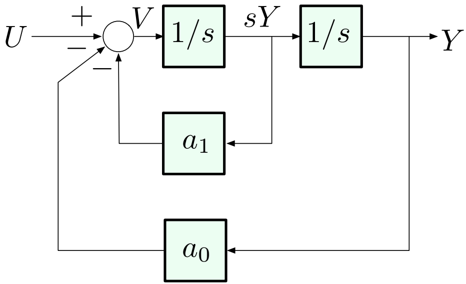

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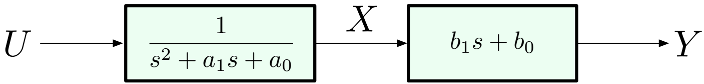

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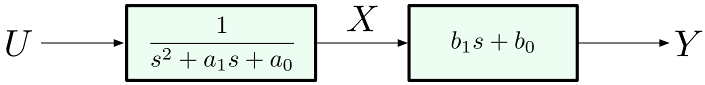

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Note: $b_0 + b_1 s$ involves *differentiation*, which we cannot implement using an all-integrator diagram. But we will see that we don't need to do it directly.

Example 3, continued

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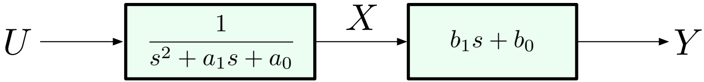

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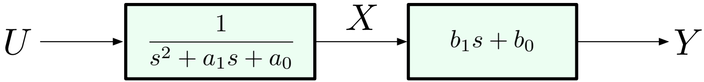

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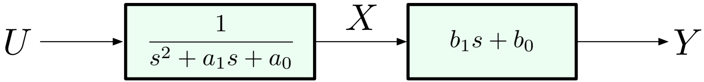

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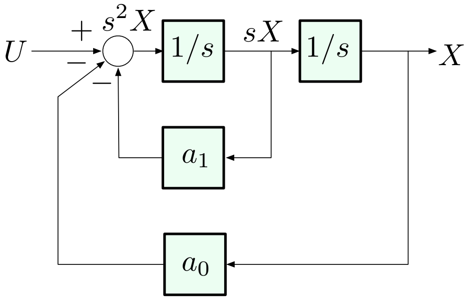

Example 3, continued

Step 3: now we notice that

$$Y(s) = b_1 sX(s) + b_0 X(s),$$

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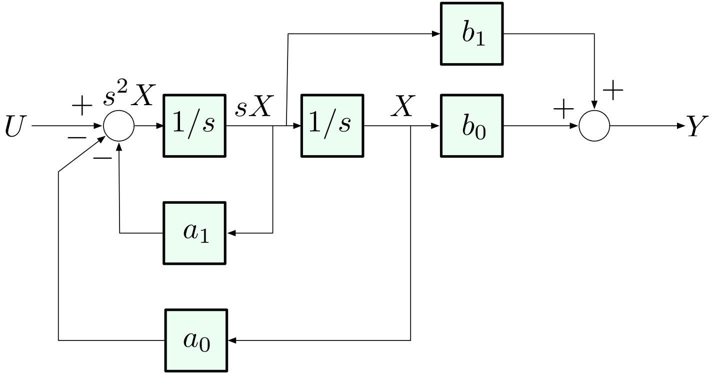
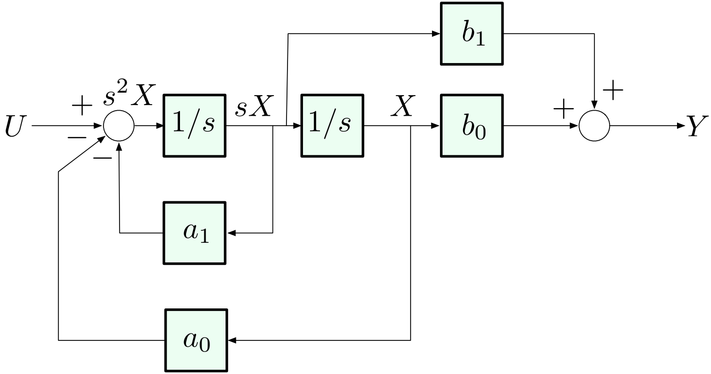


Example 3, continued

All-integrator diagram for $H(s) = \dfrac{b_1 s + b_0}{s^2 + a_1 s + a_0}$



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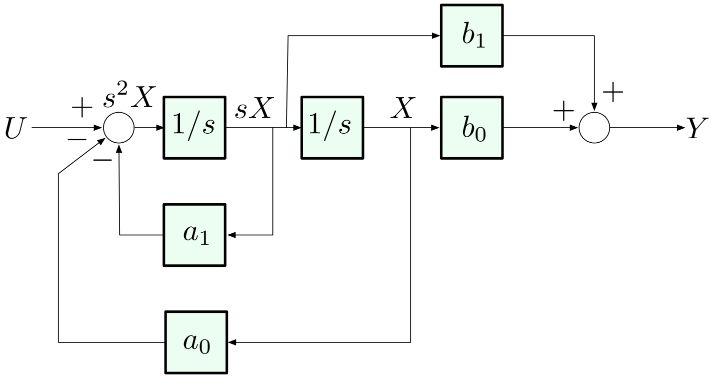
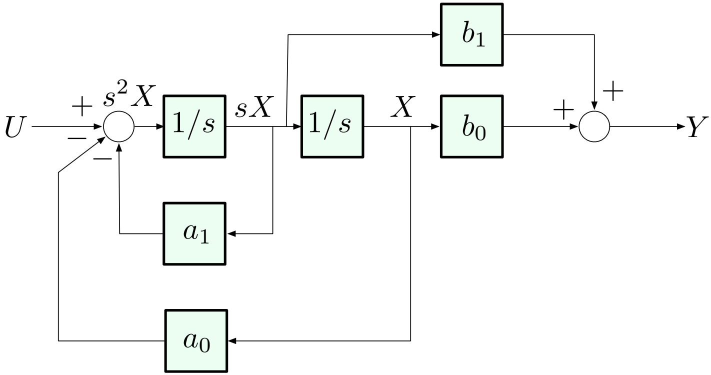


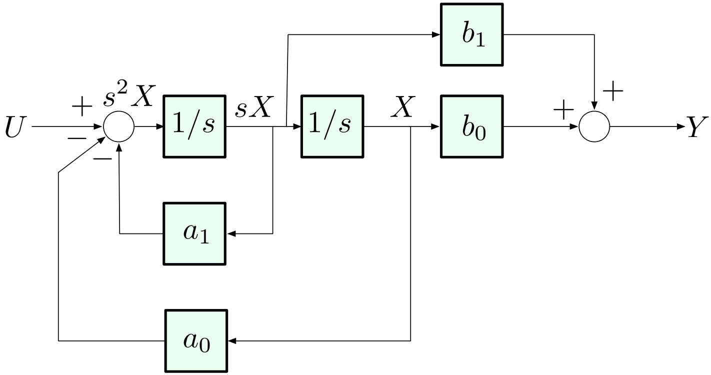
Can we write down a state-space model corresponding to this diagram?

Example 3, continued



State-space model:

Example 3, continued



State-space model:

$$s^2X = U - a_1 sX - a_0 X$$

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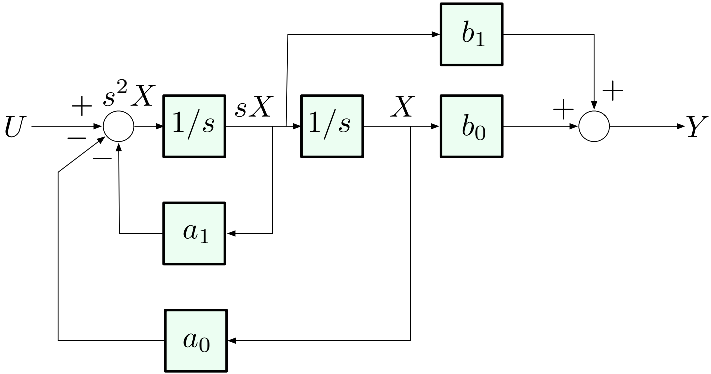

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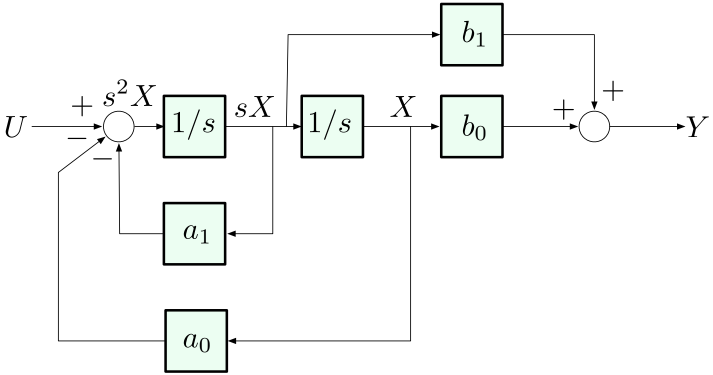

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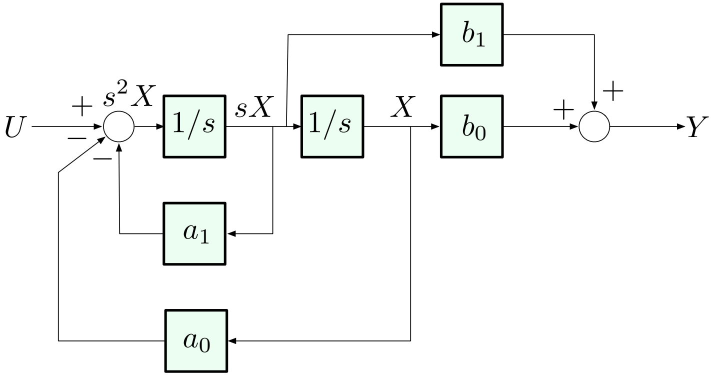

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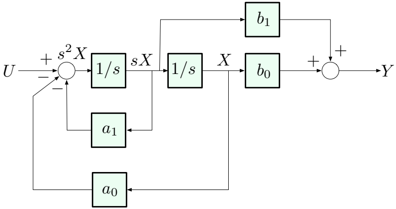
- Easily generalizes to dimension > 1
- The reason behind the name will be made clear later in the semester

Example 3, wrap-up

All-integrator diagram for $H(s) = \dfrac{b_1 s + b_0}{s^2 + a_1 s + a_0}$



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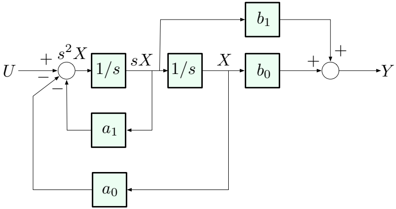

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Important: for a given $H(s)$, the diagram is *not unique*. But, once we build a diagram, the state-space equations are unique (up to coordinate transformations).

Basic System Interconnections

Now we will take this a level higher — we will talk about building complex systems from smaller blocks, without worrying about how those blocks look on the inside (they could themselves be all-integrator diagrams, etc.)

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Block diagrams describe the *flow of information*

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Series connection

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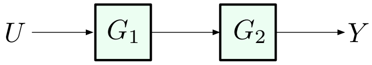

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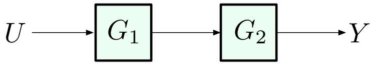

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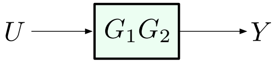

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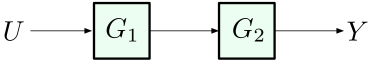

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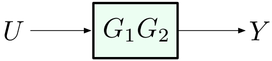

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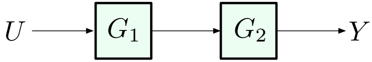

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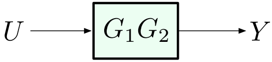

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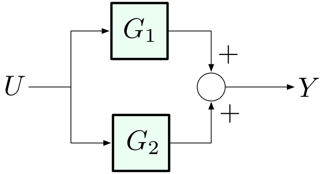

Basic System Interconnections: Series & Parallel

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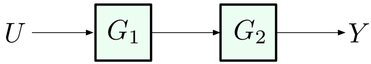

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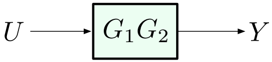

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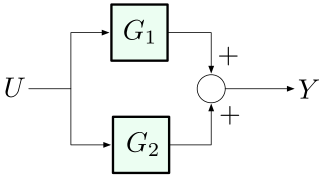
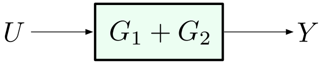
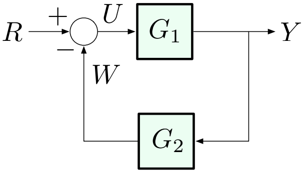


$$\frac{Y}{U} = G_1 + G_2$$

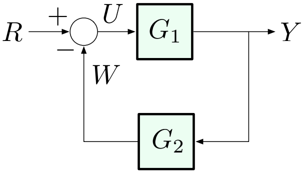


Basic System Interconnections: Negative Feedback



Find the transfer function from R (reference) to Y

Basic System Interconnections: Negative Feedback



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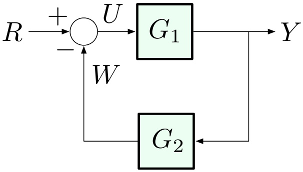

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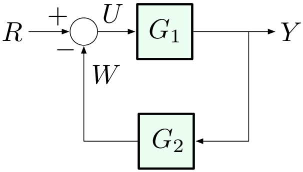

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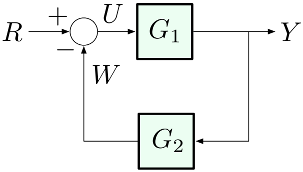

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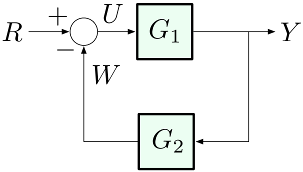

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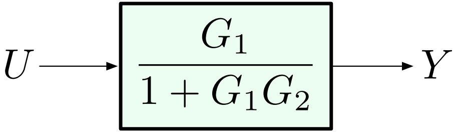

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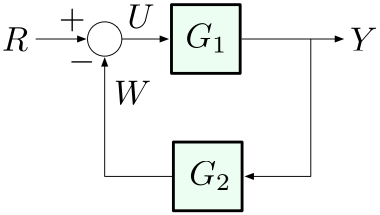

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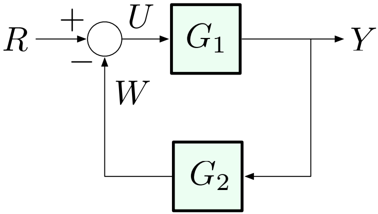

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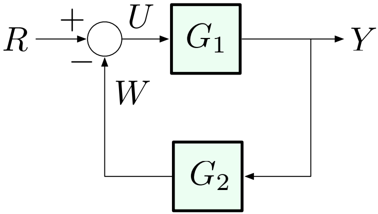

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The gain of a negative feedback loop:

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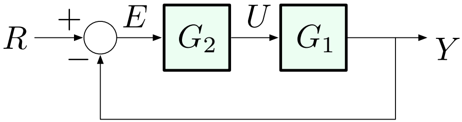
This is an important relationship, easy to derive — no need to memorize it.

Unity Feedback

Other feedback configurations are also possible:



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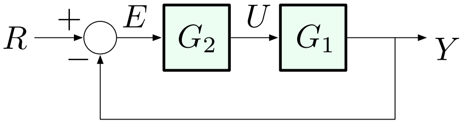

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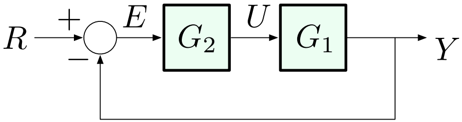

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Common structure (saw this in Lecture 1):

- R = reference
- U = control input
- Y = output
- E = error
- G_1 = plant (also denoted by P)
- G_2 = controller or compensator (also denoted by C or K)

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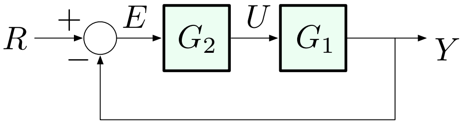

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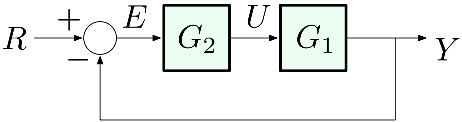

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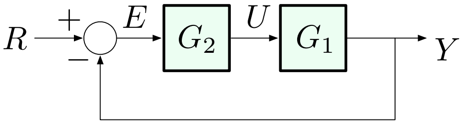

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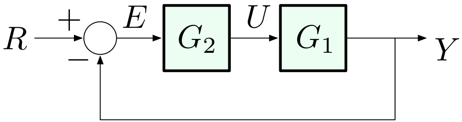

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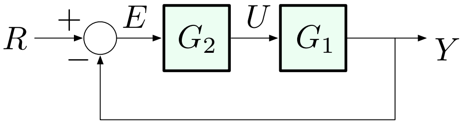

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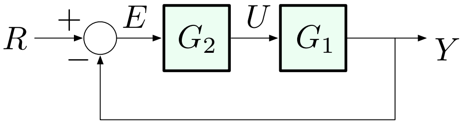

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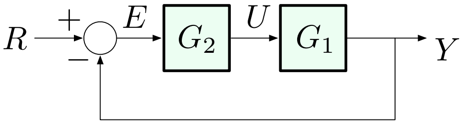

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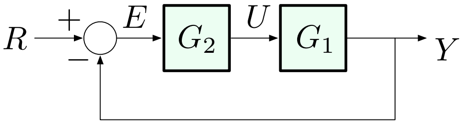

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General strategy:

- Name all the variables in the diagram
- Write down as many relationships between these variables as you can
- Learn to recognize series, parallel, and feedback interconnections
- Replace them by their equivalents
- Repeat

Prototype 2nd-Order System

So far, we have only seen transfer functions that have either real poles or purely imaginary poles:

$$\frac{1}{s+a}, \qquad \frac{1}{(s+a)(s+b)}, \qquad \frac{1}{s^2+\omega^2}$$

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Plus, you will need this for Lab 1.

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$$\text{where} \qquad \sigma = \zeta\omega_n, \; \omega_d = \omega_n\sqrt{1 - \zeta^2}$$

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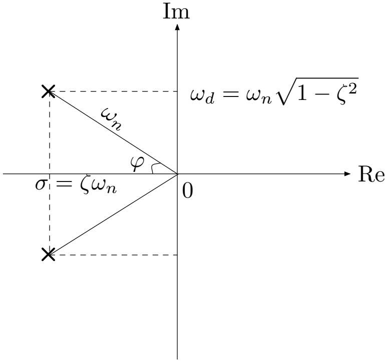

Note that

$$\begin{aligned} \sigma^2 + \omega_d^2 &= \zeta^2\omega_n^2 + \omega_n^2 - \zeta^2\omega_n^2 \\ &= \omega_n^2 \\ \cos\varphi &= \frac{\zeta\omega_n}{\omega_n} = \zeta \end{aligned}$$

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$$= 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right) \qquad \text{(table, \#21)}$$

2nd-Order Step Response

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2}$$

$$u(t) = 1(t) \qquad \longrightarrow \qquad y(t) = 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

where $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$ (damped frequency)

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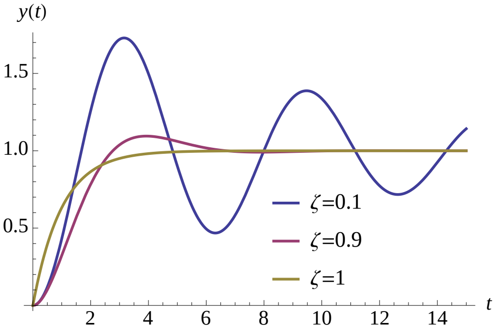

The parameter ζ is called the *damping ratio*

- $\zeta > 1$: system is overdamped
- $\zeta < 1$: system is underdamped
- $\zeta = 0$: no damping ($\omega_d = \omega_n$)