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Reading: FPE, Sections 1.1, 1.2, 2.1–2.4, 7.2, 9.2.1.
Chapter 2 has lots of cool examples of system models!!

Notation Reminder

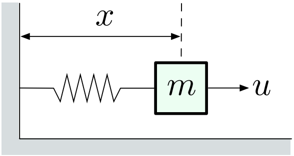
We will be looking at *dynamic systems* whose evolution *in time* is described by *differential equations* with *external inputs*.

We will not write the time variable t explicitly, so we use

$$
\begin{array}{lll}
x & \text{instead of} & x(t) \\
\dot{x} & \text{instead of} & x'(t) \text{ or } \dfrac{\mathrm{d}x}{\mathrm{d}t} \\
\ddot{x} & \text{instead of} & x''(t) \text{ or } \dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}
\end{array}
$$

etc.

Example 1: Mass-Spring System



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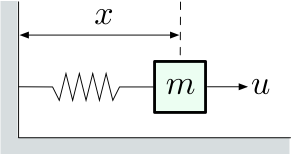

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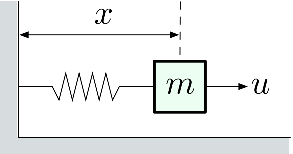

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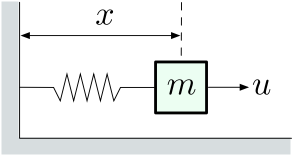

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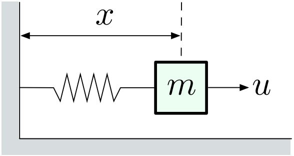

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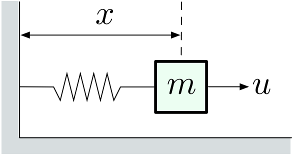

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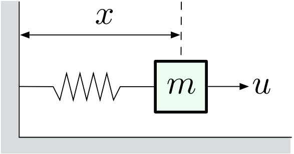

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Move $x, \dot{x}, \ddot{x}$ to the LHS, u to the RHS:

$$m\ddot{x} + \rho\dot{x} + kx = u$$

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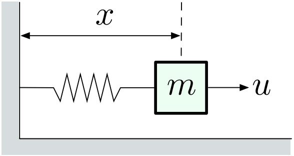

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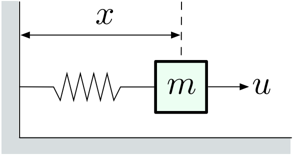

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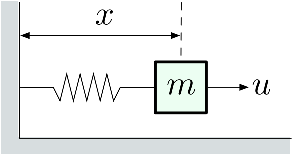

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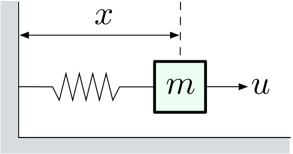

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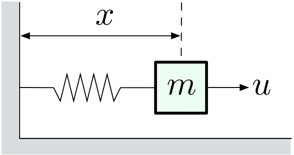

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Canonical form: convert to a *system of 1st-order ODEs*

$$\dot{x} = v \qquad \text{(definition of velocity)}$$

$$\dot{v} = -\frac{\rho}{m}v - \frac{k}{m}x + \frac{1}{m}u$$

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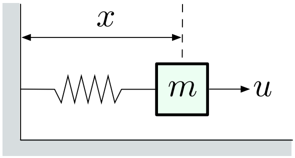

State-space model: express in *matrix form*

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\rho}{m} \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u$$

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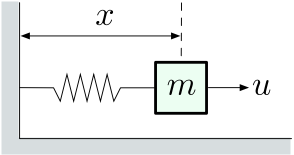

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Important: start reviewing your linear algebra ***now***!!

- matrix-vector multiplication; eigenvalues and eigenvectors; etc.

General n-Dimensional State-Space Model

$$\text{state } x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \qquad \text{input } u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \in \mathbb{R}^m$$

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$$\dot{x} = Ax + Bu$$

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Example: if we only care about (or can only measure) x_1, then

$$y = x_1 = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

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— match against $\dot{x} = Ax + Bu$

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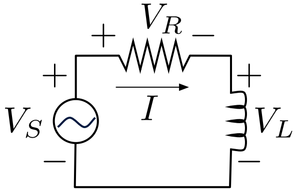

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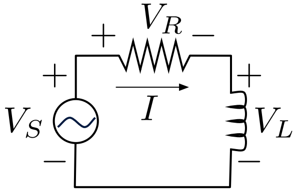

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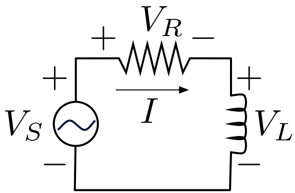

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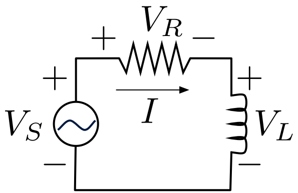

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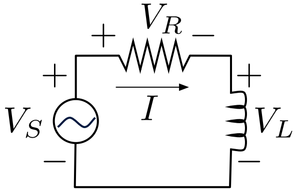

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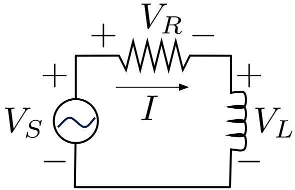

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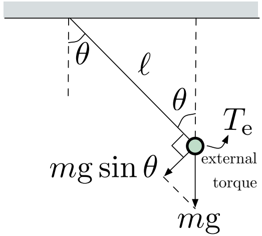

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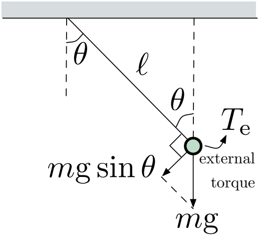

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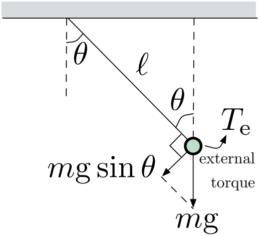

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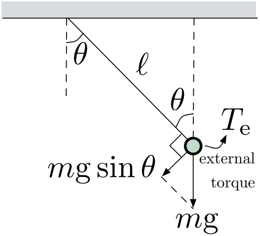

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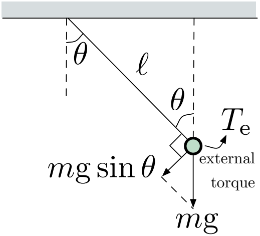

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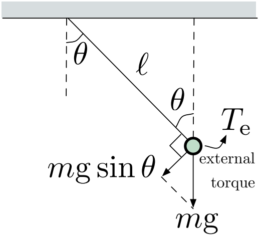

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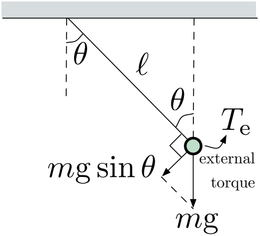

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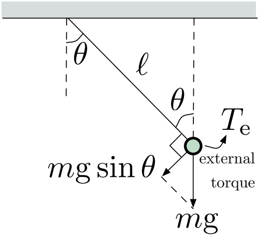

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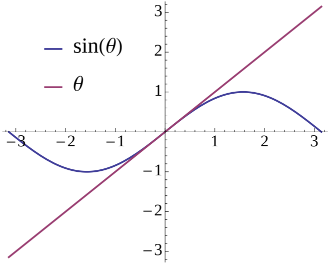

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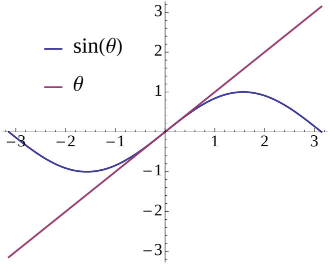

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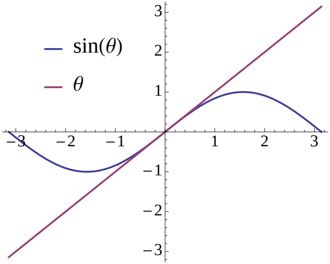

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$$\begin{pmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ -\frac{g}{\ell} & 0\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} + \begin{pmatrix}0\\ \frac{1}{m\ell^2}\end{pmatrix}T_e$$

Linearization

Taylor series expansion:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \dots$$

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$$\dot{x} = f(x, u) \qquad \text{nonlinear state-space model}$$

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Assume $x = 0, u = 0$ is an *equilibrium point*: $f(0, 0) = 0$

This means that, when the system is at rest and no control is applied, the system does not move.

Linearization

Linear approx. around $(x, u) = (0, 0)$ to all components of f:

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Important: since we have ignored the higher-order terms, this linear system is only an *approximation* that holds only for *small deviations* from equilibrium.

Example 3: Pendulum, Revisited

Original nonlinear state-space model:

$$\dot{\theta}_1 = f_1(\theta_1, \theta_2, T_e) = \theta_2 \qquad \text{— already linear}$$

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  Note that the transformation is *invertible*:

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  Any linear system *must* have an equilibrium point at $(x, u) = (0, 0)$:

$$f(x, u) = Ax + Bu \qquad f(0, 0) = A0 + B0 = 0.$$